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Intuitionistic fuzzy BCI-algebras (implicative ideals, closed implicative ideals, commutative ideals) under norms

Rasul Rasuli

Abstract. In this article, by using norms(T and C), we present the concept of intuitionistic fuzzy implicative ideals, intuitionistic fuzzy closed implicative ideals and intuitionistic fuzzy commutative ideals of BCI-algebras. Some interesting results of them are given. Characterisations of implicative ideals, closed implicative ideals and commutative ideals of BCI-algebras by using them are explored. By using intersections, direct products and homomorphisms, some interesting results are obtained.

1. Introduction

Iseki [6] introduced the idea of BCI-algebras. BCI-algebras are established from two distinct approaches as propositional calculi and set theory. Several results and properties of BCI-algebras are discussed in the work [5]. Fuzzy set theory, initially established by Zadeh [18] in 1965, was applied by several researchers to generalize some of the essential ideas of algebraic structures. Fuzzy algebraic structures play a prominent role in different domains in mathematics and other sciences. The idea of "intuitionistic fuzzy set" was first published by Atanassov [2, 3] as a generalization of the notion of fuzzy sets. Liu and et al. [7] introduced the notions of intuitionistic fuzzy implicative ideals and intuitionistic fuzzy commutative ideals of BCI-algebras and discussed their properties. Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy

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*Corresponding author

logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics. The author by using norms, investigated some properties of fuzzy algebraic structures [9]-[16]. In this paper, we define intuitionistic fuzzy implicative ideals, intuitionistic fuzzy closed implicative ideals and intuitionistic fuzzy commutative ideals of BCI-algebras respect to t-norm T and t-conorm C. Next we obtain the relation between them and implicative ideals, closed implicative ideals and commutative ideals of BCI-algebras. Finally we consider them under intersections, direct products and homomorphisms.

2. preliminaries

Definition 2.1. [17] An algebra $(X, *, 0)$ is called a BCI-algebra if it satisfies the following conditions:
(1) $((x * y) * (x * z)) * (z * y) = 0$
(2) $(x * (x * y)) * y = 0$
(3) $x * x = 0$
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$
(5) $(x * y) * z = (x * z) * y$
(6) $x * 0 = x$
(7) $0 * (x * y) = (0 * x) * (0 * y)$
(8) $0 * (0 * (x * y)) = 0 * (y * x)$
for all $x, y, z \in X$.
In a BCI-algebra, we can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.
(9) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$
(10) $(x * z) * (y * z) \leq x * y$
for all $x, y, z \in X$.

Definition 2.2. [7] A non-empty subset I of a BCI-algebra X is called an ideal of X if
(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply that $x \in I$, for all $x, y \in X$.

Definition 2.3. [7] An ideal I of a BCI-algebra X is is said to be closed if $0 * x \in I$ for all $x \in X$.

Definition 2.4. [7] A non-empty subset I of a BCI-algebra X is said to be an implicative ideal of X if it satisfies:
(1) $0 \in I$,
(2) $((x * (x * y)) * (y * x)) * z \in I$ and $z \in I$ imply $y * (y * x) \in I$, for all $x, y, z \in X$.

Definition 2.5. [7] A non-empty subset I of a BCI-algebra X is said to be a commutative ideal of X if it satisfies:

(1) $0 \in I$,
(2) $(y*(y*(x*(x*y))))*z \in I$ and $z \in I$ imply $x*(x*y) \in I$, for all $x, y, z \in X$.

Definition 2.6. A mapping $f : X \to Y$ of BCI-algebras is called a homomorphism if $f(x*y) = f(x)*f(y)$, for all $x, y \in X$.

Definition 2.7. [8] Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0,1]$. The set of all fuzzy subsets of X is called the $[0,1]$-power set of X and is denoted $[0,1]^X$. For a fixed $s \in [0,1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ and the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level of μ.

Definition 2.8. [2, 3] Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0,1) \sim 0$ and $U_X(x) = (1,0) \sim 1$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.9. [8] Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. For all $x \in X, y \in Y$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$$
$$= \begin{cases} (\sup\{\mu_A(x) | x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) | x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0,1) & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Definition 2.10. [4] A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.11. (1) Standard intersection t-norm $T_m(x,y) = \min\{x, y\}$.
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product t-norm $T_p(x,y) = xy$.

(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \le T(x,y) \le T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Definition 2.12. [4] A t-norm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(1) $C(x,0) = x$,
(2) $C(x,y) \le C(x,z)$ if $y \le z$,
(3) $C(x,y) = C(y,x)$,
(4) $C(x, C(y,z)) = C(C(x,y),z)$,
for all $x, y, z \in [0,1]$.

We say that T and C be idempotent if for all $x \in [0,1]$ we have $T(x,x) = x$ and $C(x,x) = x$.

Example 2.13. The basic t-conorms are

$$C_m(x,y) = \max\{x,y\},$$

$$C_b(x,y) = \min\{1, x+y\}$$

and

$$C_p(x,y) = x + y - xy$$

for all $x, y \in [0,1]$. S_m is standard union, C_b is bounded sum, C_p is algebraic sum.

Definition 2.14. [9] Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(X)$. Define

$$A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) : X \to [0,1]$$

as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(x))$ for all $x \in X$.

Definition 2.15. [9] Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. The cartesian product of A and B is denoted by $A \times B : X \times Y \to [0, 1]$ is defined by

$$\begin{aligned}(A \times B)(x, y) &= ((\mu_A, \nu_A) \times (\mu_B, \nu_B))(x, y)\\ &= (\mu_{A\times B}, \nu_{A\times B})(x, y)\\ &= (\mu_{A\times B}(x, y), \nu_{A\times B}(x, y))\\ &= (T(\mu_A(x), \mu_B(y)), C(\nu_A(x), \nu_B(y))),\end{aligned}$$

for all $(x, y) \in X \times Y$.

Lemma 2.1. [1] *Let C be a t-conorm and T be a t-norm. Then*

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z))$$

and

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. Main Results

Definition 3.1. We say that $A = (\mu_A, \nu_A) \in IFS(X)$ is an intuitionistic fuzzy implicative ideal of BCI-algebra X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(y * (y * x)) \geq T(\mu_A(x * (x * y) * (y * x)), \mu_A(z))$,
(3) $\nu_A(0) \leq \nu_A(x)$,
(4) $\nu_A(y * (y * x)) \leq C(\nu_A(x * (x * y) * (y * x)), \nu_A(z))$,
for all $x, y, z \in X$.
Denote by $(T, C)IFII(X)$, the set of all intuitionistic fuzzy implicative ideals of BCI-algebra X under norms(t-norm T and t-conorm C).

Example 3.2. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then $(X, *, 0)$ is a BCI-algebra. Define $A = (\mu_A, \nu_A) \in IFS(X)$ as

$$\mu_A(x) = \begin{cases} t_1 & \text{if } x = 0, 1,\\ t_2 & \text{if } x = 2,\end{cases}$$

and

$$\nu_A(x) = \begin{cases} s_1 & \text{if } x = 0, 1,\\ s_2 & \text{if } x = 2,\end{cases}$$

with $t_1 > t_2$ and $s_1 < s_2$ such that $0 < t_i + s_i < 1$ and $t_i, s_i \in [0,1]$. Let $T(a,b) = T_p(a,b) = ab$ and $C(a,b) = C_p(a,b) = a + b - ab$, for all $a, b \in [0,1]$ then $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$.

Proposition 3.1. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ if and only if the $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ is either empty or an implicative ideal of BCI-algebra X for every $s, t \in [0,1]$.*

Proof. Let $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ and $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ be not empty then for any $x \in A_{s,t}$ we have $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$ so $\mu_A(0) \geq \mu_A(x) \geq s$ and $\nu_A(0) \leq \nu_A(x) \leq t$ thus $A(0) \supseteq (s,t)$ which means that $0 \in A_{s,t}$.
Let $((x*(x*y))*(y*x))*z \in A_{s,t}$ and $z \in A_{s,t}$. Thus

$$\mu_A(y*(y*x)) \geq T(\mu_A(x*(x*y)*(y*x)), \mu_A(z)) \geq T(s,s) = s$$

and

$$\nu_A(y*(y*x)) \leq C(\nu_A(x*(x*y)*(y*x)), \nu_A(z)) \leq C(t,t) = t$$

then

$$A(y*(y*x)) = (\mu_A(y*(y*x)), \nu_A(y*(y*x))) \supseteq (s,t)$$

therefore $y*(y*x) \in A_{s,t}$. Then $A_{s,t}$ will be an implicative ideal of BCI-algebra X for every $s, t \in [0,1]$.
Conversely, let $A_{s,t}$ be not empty and be an implicative ideal of X for every $s, t \in [0,1]$. Then for any $x \in A_{s,t}$ we have $A(0) \supseteq (s,t)$ and so $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$. Let $s = T(\mu_A((x*(y*x))*(y*x)), \mu_A(z))$ and $t = C(\nu_A((x*(y*x))*(y*x)), \nu_A(z))$ with $(x*(y*x))*(y*x) \in A_{s,t}$ and $z \in A_{s,t}$. Then $y*(y*x) \in A_{s,t}$ thus

$$\mu_A(y*(y*x)) \geq s = T(\mu_A((x*(y*x))*(y*x)), \mu_A(z))$$

and

$$\nu_A(y*(y*x)) \leq t = C(\nu_A((x*(y*x))*(y*x)), \nu_A(z))$$

so $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$. □

Recall that if $A = (\mu_A, \nu_A) \in IFS(X)$ then we define $\triangle A = (\mu_A, \bar{\mu_A}) \in IFS(X)$ and $\bigtriangledown A = (\bar{\nu_A}, \nu_A) \in IFS(X)$.

Proposition 3.2. *If $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$, then $\triangle A = (\mu_A, \bar{\mu_A}) \in (T,C)IFII(X)$.*

Proof. Let $x, y, z \in X$. As $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ so $\mu_A(0) \geq \mu_A(x)$ and then $1 - \mu_A(0) \leq 1 - \mu_A(x)$. On the other hand $\bar{\mu_A}(0) \leq \bar{\mu_A}(x)$. Also since

$$\mu_A(y*(y*x)) \geq T(\mu_A(x*(x*y)*(y*x)), \mu_A(z)),$$

we have

$$1 - \mu_A(y*(y*x)) \leq 1 - T(\mu_A(x*(x*y)*(y*x)), \mu_A(z)).$$

Thus

$$\bar{\mu_A}(y*(y*x)) \le C(1-\mu_A(x*(x*y)*(y*x)), 1-\mu_A(z))$$

and consequently

$$\bar{\mu_A}(y*(y*x)) \le C(\bar{\mu_A}(x*(x*y)*(y*x)), \bar{\mu_A}(z)).$$

Therefore $\triangle A = (\mu_A, \bar{\mu_A}) \in (T,C)IFII(X)$. □

Proposition 3.3. *If* $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$, *then* $\triangledown A = (\bar{\nu_A}, \nu_A) \in (T,C)IFII(X)$.

Proof. Let $x, y, z \in X$ and $A = (\mu_A, \nu_A) \in (T,C)IFCII(X)$. Thus $\nu_A(0) \le \nu_A(x)$, so $1-\nu_A(0) \ge 1-\nu_A(x)$. Consequently $\bar{\nu_A}(0) \ge \bar{\nu_A}(x)$. Moreover as

$$\nu_A(y*(y*x)) \le C(\nu_A(x*(x*y)*(y*x)), \nu_A(z))$$

so

$$1-\nu_A(y*(y*x)) \ge 1-C(\nu_A(x*(x*y)*(y*x)), \nu_A(z))$$

which means that

$$\bar{\nu_A}(y*(y*x)) \ge T(1-\nu_A(x*(x*y)*(y*x)), 1-\nu_A(z)).$$

This means that

$$\bar{\nu_A}(y*(y*x)) \ge T(\bar{\nu_A}(x*(x*y)*(y*x)), \bar{\nu_A}(z)).$$

Hence $\triangledown A = (\bar{\nu_A}, \nu_A) \in (T,C)IFII(X)$. □

Proposition 3.4. $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ *if and only if* $\triangle A = (\mu_A, \bar{\mu_A}) \in (T,C)IFII(X)$ *and* $\triangledown A = (\bar{\nu_A}, \nu_A) \in (T,C)IFII(X)$.

Proof. Use Proposition 3.2 and Proposition 3.3. □

Proposition 3.5. *Let* $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ *and* $B = (\mu_B, \nu_B) \in (T,C)IFII(X)$. *Then* $A \cap B \in (T,C)IFII(X)$.

Proof. Let $x, y, z \in X$. Then

(1)

$$\mu_{A\cap B}(0) = T(\mu_A(0), \mu_B(0)) \ge T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$$

thus

$$\mu_{A\cap B}(0) \ge \mu_{A\cap B}(x).$$

(2)

$$\begin{aligned}
&\mu_{A\cap B}(y*(y*x)) = T(\mu_A(y*(y*x)), \mu_B(y*(y*x))) \\
&\ge T(T(\mu_A(x*(x*y)*(y*x)), \mu_A(z)), T(\mu_B(x*(x*y)*(y*x)), \mu_B(z))) \\
&= T(T(\mu_A(x*(x*y)*(y*x)), \mu_B(x*(x*y)*(y*x)), T(\mu_A(z), \mu_B(z))) \\
&= T(\mu_{A\cap B}(x*(x*y)*(y*x), \mu_{A\cap B}(z))
\end{aligned}$$

so

$$\mu_{A\cap B}(y*(y*x)) \geq T(\mu_{A\cap B}(x*(x*y)*(y*x), \mu_{A\cap B}(z)).$$

(3)

$$\begin{aligned}
&\nu_{A\cap B}(y*(y*x)) = C(\nu_A(y*(y*x)), \nu_B(y*(y*x)))\\
&\leq C(C(\nu_A(x*(x*y)*(y*x)), \nu_A(z)), C(\nu_B(x*(x*y)*(y*x)), \nu_B(z)))\\
&= C(C(\nu_A(x*(x*y)*(y*x)), \nu_B(x*(x*y)*(y*x)), C(\nu_A(z), \nu_B(z)))\\
&= C(\nu_{A\cap B}(x*(x*y)*(y*x), \nu_{A\cap B}(z)).
\end{aligned}$$

So

$$\nu_{A\cap B}(y*(y*x)) \leq C(\nu_{A\cap B}(x*(x*y)*(y*x), \nu_{A\cap B}(z)).$$

Now (1)-(4) give us that $A\cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in (T,C)IFII(X)$. □

Proposition 3.6. *Let $A = (\mu_A, \nu_A) \in (T,C)IFII(X)$ and $B = (\mu_B, \nu_B) \in (T,C)IFII(Y)$. Then $A \times B \in (T,C)IFII(X\times Y)$.*

Proof. Let $(x,y) \in X \times Y$. Then

$$\mu_{A\times B}(0,0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(y)) = \mu_{A\times B}(x,y)$$

and

$$\nu_{A\times B}(0,0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(y)) = \nu_{A\times B}(x,y).$$

Also let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in X\times Y$. Now

$$\begin{aligned}
&\mu_{A\times B}((y_1,y_2)*((y_1,y_2)*(x_1,x_2))) = \mu_{A\times B}((y_1,y_2)*(y_1*x_1, y_2*x_2))\\
&= \mu_{A\times B}(y_1*(y_1*x_1), y_2*(y_2*x_2))\\
&= T(\mu_A(y_1*(y_1*x_1)), \mu_B(y_2*(y_2*x_2)))\\
&\geq T(T(\mu_A(x_1*(x_1*y_1)*(y_1*x_1)), \mu_A(z_1)), T(\mu_B(x_2*(x_2*y_2)*(y_2*x_2)), \mu_B(z_2)))\\
&= T(T(\mu_A(x_1*(x_1*y_1)*(y_1*x_1)), \mu_B(x_2*(x_2*y_2)*(y_2*x_2))), T(\mu_A(z_1), \mu_B(z_2)))\\
&= T(\mu_{A\times B}(x_1*(x_1*y_1)*(y_1*x_1), x_2*(x_2*y_2)*(y_2*x_2)), \mu_{A\times B}(z_1,z_2))\\
&= T(\mu_{A\times B}((x_1,x_2)*((x_1,x_2)*(y_1,y_2))*((y_1,y_2)*(x_1,x_2))), \mu_{A\times B}(z_1,z_2)).
\end{aligned}$$

So

$$\begin{aligned}
&\mu_{A\times B}((y_1,y_2)*((y_1,y_2)*(x_1,x_2)))\\
&\geq T(\mu_{A\times B}((x_1,x_2)*((x_1,x_2)*(y_1,y_2))*((y_1,y_2)*(x_1,x_2))), \mu_{A\times B}(z_1,z_2)).
\end{aligned}$$

Also

$$\nu_{A\times B}((y_1,y_2)*((y_1,y_2)*(x_1,x_2))) = \nu_{A\times B}((y_1,y_2)*(y_1*x_1,y_2*x_2))$$
$$= \nu_{A\times B}(y_1*(y_1*x_1),y_2*(y_2*x_2)) = C(\nu_A(y_1*(y_1*x_1)),\nu_B(y_2*(y_2*x_2)))$$
$$\leq C(C(\nu_A(x_1*(x_1*y_1)*(y_1*x_1)),\nu_A(z_1)),C(\nu_B(x_2*(x_2*y_2)*(y_2*x_2)),\nu_B(z_2)))$$
$$= C(C(\nu_A(x_1*(x_1*y_1)*(y_1*x_1)),\nu_B(x_2*(x_2*y_2)*(y_2*x_2))),C(\nu_A(z_1),\nu_B(z_2)))$$
$$= C(\nu_{A\times B}(x_1*(x_1*y_1)*(y_1*x_1),x_2*(x_2*y_2)*(y_2*x_2)),\mu_{A\times B}(z_1,z_2))$$
$$= C(\nu_{A\times B}((x_1,x_2)*((x_1,x_2)*(y_1,y_2))*((y_1,y_2)*(x_1,x_2))),\nu_{A\times B}(z_1,z_2)).$$

Thus

$$\nu_{A\times B}((y_1,y_2)*((y_1,y_2)*(x_1,x_2)))$$
$$\leq C(\nu_{A\times B}((x_1,x_2)*((x_1,x_2)*(y_1,y_2))*((y_1,y_2)*(x_1,x_2))),\nu_{A\times B}(z_1,z_2)).$$

Therefore,

$$A\times B = (\mu_{A\times B},\nu_{A\times B}) \in (T,C)IFII(X\times Y).$$

□

Proposition 3.7. *If $A = (\mu_A,\nu_A) \in (T,C)IFII(X)$ and $\varphi : X \to Y$ is a homomorphism of BCI-algebras, then $\varphi(A) \in (T,C)IFII(Y)$.*

Proof. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu_A)(0) = \sup\{\mu_A(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu_A)(y)$$

thus

$$\varphi(\mu_A)(0) \geq \varphi(\mu_A)(y)$$

and

$$\varphi(\nu_A)(0) = \inf\{\nu_A(0) \mid 0 \in X, \varphi(0) = 0\} \leq \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\nu_A)(y)$$

then

$$\varphi(\nu_A)(0) \leq \varphi(\nu_A)(y).$$

Also let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1,2,3$. Then

$$\varphi(\mu_A)(y_1*(y_1*y_2))$$
$$= \sup\{\mu_A(x_1*(x_1*x_2)) | x_1*(x_1*x_2) \in X, \varphi(x_1*(x_1*x_2)) = y_1*(y_1*y_2)\}$$
$$\geq \sup\{T(\mu_A(x_1*(x_2*x_1)*(x_1*x_2)),\mu_A(x_3)) | x_i \in X, \varphi(x_i) = y_i\}$$
$$= T(\sup\{\mu_A(x_1*(x_2*x_1)*(x_1*x_2)) | x_i \in X, \varphi(x_1*(x_2*x_1)*(x_1*x_2)) = y_1*(y_2*y_1)*(y_1*y_2)\}, \sup\{\mu_A(x_3) | x_3 \in X, \varphi(x_3) = y_3\})$$
$$= T(\varphi(\mu_A)(y_1*(y_2*y_1)*(y_1*y_2)),\varphi(\mu_A)(y_3)).$$

Thus

$$\varphi(\mu_A)(y_1*(y_1*y_2)) \geq T(\varphi(\mu_A)(y_1*(y_2*y_1)*(y_1*y_2)),\varphi(\mu_A)(y_3)).$$

Also

$$\begin{aligned}
&\varphi(\nu_A)(y_1 * (y_1 * y_2))\\
&= \inf\{\nu_A(x_1 * (x_1 * x_2)) | x_1 * (x_1 * x_2) \in X, \varphi(x_1 * (x_1 * x_2)) = y_1 * (y_1 * y_2)\}\\
&\leq \inf\{C(\nu_A(x_1 * (x_2 * x_1) * (x_1 * x_2)), \nu_A(x_3)) | x_i \in X, \varphi(x_i) = y_i\}\\
&= C(\inf\{\nu_A(x_1 * (x_2 * x_1) * (x_1 * x_2)) \mid x_i \in X, \varphi(x_1 * (x_2 * x_1) * (x_1 * x_2)) =\\
&\quad = y_1 * (y_2 * y_1) * (y_1 * y_2)\}, \inf\{\nu_A(x_3) \mid x_3 \in X, \varphi(x_3) = y_3\})\\
&= C(\varphi(\nu_A)(y_1 * (y_2 * y_1) * (y_1 * y_2)), \varphi(\nu_A)(y_3))
\end{aligned}$$

Then

$$\varphi(\nu_A)(y_1 * (y_1 * y_2)) \leq C(\varphi(\nu_A)(y_1 * (y_2 * y_1) * (y_1 * y_2)), \varphi(\nu_A)(y_3)).$$

Therefore $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in (T, C)IFII(Y)$. □

Proposition 3.8. *If $B = (\mu_B, \nu_B) \in (T, C)IFII(Y)$ and $\varphi : X \to Y$ is a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFII(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\mu_B)(0) = \mu_B(\varphi(0)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(0) = \nu_B(\varphi(0)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

Let $x_1, x_2, x_3 \in X$. Now

$$\begin{aligned}
\varphi^{-1}(\mu_B)(x_1 * (x_1 * x_2))) &= \mu_B(\varphi(x_1 * (x_1 * x_2))))\\
&= \mu_B(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2)))\\
&\geq T(\mu_B(\varphi(x_1) * (\varphi(x_2) * \varphi(x_1)) * (\varphi(x_1) * \varphi(x_2)), \mu_B(\varphi(x_3)))\\
&= T(\mu_B(\varphi(x_1 * (x_2 * x_1) * (x_1 * x_2)), \mu_B(\varphi(x_3)))\\
&= T(\varphi^{-1}(\mu_B)(x_1 * (x_2 * x_1) * (x_1 * x_2)), \varphi^{-1}(\mu_B)(x_3)).
\end{aligned}$$

Then

$$\varphi^{-1}(\mu_B)(x_1 * (x_1 * x_2))) \geq T(\varphi^{-1}(\mu_B)(x_1 * (x_2 * x_1) * (x_1 * x_2)), \varphi^{-1}(\mu_B)(x_3)).$$

Also

$$\begin{aligned}
\varphi^{-1}(\nu_B)(x_1 * (x_1 * x_2))) &= \nu_B(\varphi(x_1 * (x_1 * x_2))))\\
&= \nu_B(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2)))\\
&\leq C(\nu_B(\varphi(x_1) * (\varphi(x_2) * \varphi(x_1)) * (\varphi(x_1) * \varphi(x_2)), \nu_B(\varphi(x_3)))\\
&= C(\nu_B(\varphi(x_1 * (x_2 * x_1) * (x_1 * x_2)), \nu_B(\varphi(x_3)))\\
&= C(\varphi^{-1}(\nu_B)(x_1 * (x_2 * x_1) * (x_1 * x_2)), \varphi^{-1}(\nu_B)(x_3))
\end{aligned}$$

Thus

$$\varphi^{-1}(\nu_B)(x_1*(x_1*x_2))) \leq C(\varphi^{-1}(\nu_B)(x_1*(x_2*x_1)*(x_1*x_2)), \varphi^{-1}(\nu_B)(x_3)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in (T,C)IFII(X)$. □

Definition 3.3. We say that $A = (\mu_A, \nu_A) \in IFS(X)$ is an intuitionistic fuzzy closed implicative ideal of BCI-algebra X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0*x) \geq \mu_A(x)$,
(2) $\mu_A(y*(y*x)) \geq T(\mu_A(x*(x*y)*(y*x)), \mu_A(z))$,
(3) $\nu_A(0*x) \leq \nu_A(x)$,
(4) $\nu_A(y*(y*x)) \leq C(\nu_A(x*(x*y)*(y*x)), \nu_A(z))$,
for all $x, y, z \in X$.

Denote by $A = (\mu_A, \nu_A) \in (T,C)IFCII(X)$, the set of all intuitionistic fuzzy closed implicative ideals of X under norms(t-norm T and t-conorm C).

Proposition 3.9. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T,C)IFCPII(X)$ if and only if the $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ is either empty or a closed implicative ideal of BCI-algebra X for every $s, t \in [0,1]$.*

Proof. Let $A = (\mu_A, \nu_A) \in (T,C)IFCII(X)$ and $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ be not empty, then for any $x \in A_{s,t}$, we have $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$ so $\mu_A(0*x) \geq \mu_A(x) \geq s$ and $\nu_A(0*x) \leq \nu_A(x) \leq t$ thus $A(0*x) \supseteq (s,t)$ which means that $0*x \in A_{s,t}$.

Also let $x*(x*y)*(y*x) \in A_{s,t}$ and $z \in A_{s,t}$. Then

$$\mu_A(y*(y*x)) \geq T(\mu_A(x*(x*y)*(y*x)), \mu_A(z)) \geq T(s,s) = s$$

and

$$\nu_A(y*(y*x)) \leq C(\nu_A(x*(x*y)*(y*x)), \nu_A(z)) \leq C(t,t) = t.$$

Thus $y*(y*x) \in A_{s,t}$. Then $A_{s,t}$ is a closed implicative ideal of X for every $s, t \in [0,1]$.

Conversely, let $A_{s,t}$ be not empty and be a closed implicative ideal of X for every $s, t \in [0,1]$. Then for any $x \in A_{s,t}$ we have $0*x \in A_{s,t}$ then $A(0*x) \supseteq (s,t)$ and so $\mu_A(0*x) \geq s$ and $\nu_A(0*x) \leq t$. Let $s = T(\mu_A(x*(x*y)*(y*x)), \mu_A(z))$ and $t = C(\nu_A(x*(x*y)*(y*x)), \nu_A(z))$ with $x*(x*y)*(y*x) \in A_{s,t}$ and $z \in A_{s,t}$. Thus $y*(y*x) \in A_{s,t}$. Therefore

$$\mu_A(y*(y*x)) \geq s = T(\mu_A(x*(x*y)*(y*x)), \mu_A(z))$$

and

$$\nu_A(y*(y*x)) \leq t = C(\nu_A(x*(x*y)*(y*x)), \nu_A(z)).$$

So $A = (\mu_A, \nu_A) \in (T,C)IFCII(X)$. □

Proposition 3.10. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. If $A = (\mu_A, \nu_A) \in (T, C)IFCII(X)$, then $J = \{x \in X : A(x) = A(0)\}$ is either empty or a closed positive implicative ideal of BCI-algebra X.*

Proof. Let $x, y, z \in X$. As $0 \in J$ so J will be not empty. Let $x \in J$ then

$$\mu_A(0 * x) \geq \mu_A(x) = \mu_A(0) \geq \mu_A(0 * x)$$

and

$$\nu_A(0 * x) \leq \nu_A(x) = \nu_A(0) \leq \nu_A(0 * x).$$

Thus $\mu_A(0 * x) = \mu_A(0)$ and $\nu_A(0 * x) = \nu_A(0)$ which mean that

$$A(0 * x) = (\mu_A(0 * x), \nu_A(0 * x)) = (\mu_A(0), \nu_A(0)) = A(0).$$

Then $0 * x \in J$. Now let $x * (x * y) * (y * x) \in J$ and $z \in J$. Then

$$\begin{aligned}\mu_A(y * (y * x)) &\geq T(\mu_A(x * (x * y) * (y * x)), \mu_A(z)) \\ &= T(\mu_A(0), \mu_A(0)) = \mu_A(0) \geq \mu_A(y * (y * x))\end{aligned}$$

and

$$\begin{aligned}\nu_A(y * (y * x)) &\leq C(\nu_A(x * (x * y) * (y * x)), \nu_A(z)) \\ &= C(\nu_A(0), \nu_A(0)) = \nu_A(0) \leq \nu_A(y * (y * x)).\end{aligned}$$

Therefore $\mu_A(y * (y * x)) = \mu_A(0)$ and $\nu_A(y * (y * x)) = \nu_A(0)$ which yield

$$A(y * (y * x)) = (\mu_A(y * (y * x)), \nu_A(y * (y * x))) = (\mu_A(0), \nu_A(0)) = A(0).$$

Consequently $J = \{x \in X : A(x) = A(0)\}$ will be a closed implicative ideal of BCI-algebra X. □

The following propositions are obvious and we omit the proof of them.

Proposition 3.11. *Let $A = (\mu_A, \nu_A) \in (T, C)IFCII(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFCII(X)$. Then $A \cap B \in (T, C)IFCPII(X)$.*

Proposition 3.12. *Let $A = (\mu_A, \nu_A) \in (T, C)IFCII(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFCII(Y)$. Then $A \times B \in (T, C)IFCII(X \times Y)$.*

Proposition 3.13. *If $A = (\mu_A, \nu_A) \in (T, C)IFCII(X)$ and $\varphi : X \to Y$ is a homomorphism of BCI-algebras, then $\varphi(A) \in (T, C)IFCII(Y)$.*

Proposition 3.14. *If $B = (\mu_B, \nu_B) \in (T, C)IFCII(Y)$ and $\varphi : X \to Y$ is a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFCII(X)$.*

Definition 3.4. We say that $A = (\mu_A, \nu_A) \in IFS(X)$ is an intuitionistic fuzzy commutative ideal of BCI-algebra X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(x * (x * y)) \geq T(\mu_A(y * (y * (x * (x * y)))), \mu_A(z))$,

(3) $\nu_A(0) \leq \nu_A(x)$,
(4) $\nu_A(x*(x*y)) \leq C(\nu_A(y*(y*(x*(x*y)))), \nu_A(z))$,
for all $x, y, z \in X$.
Denote by $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$, the set of all intuitionistic fuzzy commutative ideals of BCI-algebra X under norms(t-norm T and t-conorm C).

Example 3.5. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X, *, 0)$ is a BCI-algebra. Define $A = (\mu_A, \nu_A) \in IFS(X)$ as

$$\mu_A(x) = \begin{cases} 0.75 & \text{if } x = 0, 3, \\ 0.45 & \text{if } x = 1, 2, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.15 & \text{if } x = 0, 3, \\ 0.35 & \text{if } x = 1, 2, \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$, for all $a, b \in [0, 1]$ then $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$.

Proposition 3.15. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$ if and only if the $A_{s,t} = \{x \in X : A(x) \supseteq (s, t)\}$ is either empty or a commutative ideal of BCI-algebra X for every $s, t \in [0, 1]$.*

Proof. Let $A = (\mu_A, \nu_A) \in (T, C)IFCII(X)$ and $A_{s,t} = \{x \in X : A(x) \supseteq (s, t)\}$ be not empty then for any $x \in A_{s,t}$ we have $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$ so $\mu_A(0) \geq \mu_A(x) \geq s$ and $\nu_A(0) \leq \nu_A(x) \leq t$ thus $A(0) \supseteq (s, t)$ which means that $0 \in A_{s,t}$. Also let $y*(y*(x*(x*y))) \in A_{s,t}$ and $z \in A_{s,t}$. Then

$$\mu_A(x*(x*y)) \geq T(\mu_A(y*(y*(x*(x*y)))), \mu_A(z)) \geq T(s, s) = s$$

and

$$\nu_A(x*(x*y)) \leq C(\nu_A(y*(y*(x*(x*y)))), \nu_A(z)) \leq C(t, t) = t.$$

Thus $x*(x*y) \in A_{s,t}$. Then $A_{s,t}$ is a commutative ideal of X for every $s, t \in [0, 1]$.

Conversely, let $A_{s,t}$ be not empty and be a commutative ideal of X for every $s, t \in [0, 1]$. As $0 \in A_{s,t}$ so for any $x \in A_{s,t}$ we have then $A(0) \supseteq (s, t)$ and so $\mu_A(0) \geq \mu_A(x) \geq$ and $\nu_A(0) \leq \nu_A(x)$. Let $s = T(\mu_A(y*(y*(x*(x*y)))), \mu_A(z))$ and $t = C(\nu_A(y*(y*(x*(x*y)))), \nu_A(z))$ with $y*(y*(x*(x*y))) \in A_{s,t}$ and $z \in A_{s,t}$. Thus $x*(x*y) \in A_{s,t}$. Therefore

$$\mu_A(x*(x*y)) \geq s = T(\mu_A(y*(y*(x*(x*y)))), \mu_A(z))$$

and

$$\nu_A(y*(y*x)) \leq t = C(\nu_A(y*(y*(x*(x*y)))), \nu_A(z))$$

so $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$. □

Proposition 3.16. *Let $A = (\mu_A, \nu_A) \in (T, C)IFII(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFCI(X)$. Then $A \cap B \in (T, C)IFCI(X)$.*

Proof. Let $x, y, z \in X$. Then

(1)

$$\mu_{A\cap B}(0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$$

thus

$$\mu_{A\cap B}(0) \geq \mu_{A\cap B}(x).$$

(2)

$$\nu_{A\cap B}(0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(x)) = \nu_{A\cap B}(x)$$

then

$$\nu_{A\cap B}(0) \leq \nu_{A\cap B}(x).$$

(3)

$$\begin{aligned}
&\mu_{A\cap B}(x*(x*y)) = T(\mu_A(x*(x*y)), \mu_B(x*(x*y))) \\
&\geq T(T(\mu_A(y*(y*(x*(x*y)))), \mu_A(z)), T(\mu_B(y*(y*(x*(x*y)))), \mu_B(z))) \\
&= T(T(\mu_A(y*(y*(x*(x*y)))), \mu_B(y*(y*(x*(x*y))))), T(\mu_A(z), \mu_B(z))) \\
&= T(\mu_{A\cap B}(y*(y*(x*(x*y)))), \mu_{A\cap B}(z)).
\end{aligned}$$

So

$$\mu_{A\cap B}(x*(x*y)) \geq T(\mu_{A\cap B}(y*(y*(x*(x*y)))), \mu_{A\cap B}(z)).$$

(4)

$$\begin{aligned}
&\nu_{A\cap B}(x*(x*y)) = C(\nu_A(x*(x*y)), \nu_B(x*(x*y))) \\
&\leq C(C(\nu_A(y*(y*(x*(x*y)))), \nu_A(z)), C(\nu_B(y*(y*(x*(x*y)))), \nu_B(z))) \\
&= C(C(\nu_A(y*(y*(x*(x*y)))), \nu_B(y*(y*(x*(x*y))))), C(\nu_A(z), \nu_B(z))) \\
&= C(\nu_{A\cap B}(y*(y*(x*(x*y)))), \nu_{A\cap B}(z)).
\end{aligned}$$

So

$$\nu_{A\cap B}(x*(x*y)) \leq C(\nu_{A\cap B}(x*(x*y)*(y*x), \nu_{A\cap B}(z)).$$

Now (1)-(4) give us that $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in (T, C)IFCI(X)$. □

Proposition 3.17. *Let $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFCI(Y)$. Then $A \times B \in (T, C)IFCI(X \times Y)$.*

Proof. Let $(x, y) \in X \times Y$. Then

$$\mu_{A\times B}(0,0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(y)) = \mu_{A\times B}(x, y)$$

and

$$\nu_{A\times B}(0,0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(y)) = \nu_{A\times B}(x, y).$$

Also let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$. Now

$$\begin{aligned}
&\mu_{A\times B}((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))) = \mu_{A\times B}((x_1, x_2) * (x_1 * y_1, x_2 * y_2)) \\
&= \mu_{A\times B}(x_1 * (x_1 * y_1), x_2 * (x_2 * y_2)) = T(\mu_A(x_1 * (x_1 * y_1)), \mu_B(x_2 * (x_2 * y_2))) \\
&\geq T(T(\mu_A(y_1 * (y_1 * (x_1 * (x_1 * y_1)))), \mu_A(z_1)), T(\mu_B(y_2 * (y_2 * (x_2 * (x_2 * y_2)))), \mu_B(z_2))) \\
&= T(T(\mu_A(y_1 * (y_1 * (x_1 * (x_1 * y_1)))), \mu_B(y_2 * (y_2 * (x_2 * (x_2 * y_2))))), T(\mu_A(z_1), \mu_B(z_2))) \\
&= T(\mu_{A\times B}(y_1 * (y_1 * (x_1 * (x_1 * y_1))), y_2 * (y_2 * (x_2 * (x_2 * y_2)))), \mu_{A\times B}(z_1, z_2)) \\
&= T(\mu_{A\times B}((y_1, y_2) * ((y_1, y_2) * ((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))))), \mu_{A\times B}(z_1, z_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mu_{A\times B}((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))) \\
&\geq T(\mu_{A\times B}((y_1, y_2) * ((y_1, y_2) * ((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))))), \mu_{A\times B}(z_1, z_2)).
\end{aligned}$$

Also

$$\begin{aligned}
&\nu_{A\times B}((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))) = \nu_{A\times B}((x_1, x_2) * (x_1 * y_1, x_2 * y_2)) \\
&= \nu_{A\times B}(x_1 * (x_1 * y_1), x_2 * (x_2 * y_2)) \\
&= C(\nu_A(x_1 * (x_1 * y_1)), \nu_B(x_2 * (x_2 * y_2))) \\
&\leq C(C(\nu_A(y_1 * (y_1 * (x_1 * (x_1 * y_1)))), \nu_A(z_1)), C(\nu_B(y_2 * (y_2 * (x_2 * (x_2 * y_2)))), \nu_B(z_2))) \\
&= C(C(\nu_A(y_1 * (y_1 * (x_1 * (x_1 * y_1)))), \mu_B(y_2 * (y_2 * (x_2 * (x_2 * y_2))))), T(\mu_A(z_1), \nu_B(z_2))) \\
&= C(\nu_{A\times B}(y_1 * (y_1 * (x_1 * (x_1 * y_1))), y_2 * (y_2 * (x_2 * (x_2 * y_2)))), \nu_{A\times B}(z_1, z_2)) \\
&= C(\nu_{A\times B}((y_1, y_2) * ((y_1, y_2) * ((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))))), \nu_{A\times B}(z_1, z_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
&\nu_{A\times B}((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))) \\
&\leq C(\nu_{A\times B}((y_1, y_2) * ((y_1, y_2) * ((x_1, x_2) * ((x_1, x_2) * (y_1, y_2))))), \nu_{A\times B}(z_1, z_2)).
\end{aligned}$$

Therefore

$$A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in (T, C)IFCI(X \times Y).$$

□

Proposition 3.18. *If $A = (\mu_A, \nu_A) \in (T, C)IFCI(X)$ and $\varphi : X \to Y$ is a homomorphism of BCI-algebras, then $\varphi(A) \in (T, C)IFCI(Y)$.*

PROOF. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\begin{aligned}\varphi(\mu_A)(0) &= \sup\{\mu_A(0)|0 \in X, \varphi(0) = 0\}\\ &\geq \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}\\ &= \varphi(\mu_A)(y).\end{aligned}$$

Thus

$$\varphi(\mu_A)(0) \geq \varphi(\mu_A)(y)$$

and

$$\begin{aligned}\varphi(\nu_A)(0) &= \inf\{\nu_A(0) \mid 0 \in X, \varphi(0) = 0\}\\ &\leq \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}\\ &= \varphi(\nu_A)(y).\end{aligned}$$

Then

$$\varphi(\nu_A)(0) \leq \varphi(\nu_A)(y).$$

Also let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2, 3$. Then

$$\begin{aligned}&\varphi(\mu_A)(y_1 * (y_1 * y_2))\\ &= \sup\{\mu_A(x_1 * (x_1 * x_2))|x_1 * (x_1 * x_2) \in X, \varphi(x_1 * (x_1 * x_2)) = y_1 * (y_1 * y_2)\}\\ &\geq \sup\{T(\mu_A(x_2 * (x_2 * (x_1 * (x_1 * x_2)))), \mu_A(x_3))|x_i \in X, \varphi(x_i) = y_i\}\\ &= T(\sup\{\mu_A(x_2 * (x_2 * (x_1 * (x_1 * x_2))))|x_i \in X, \varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2)))) =\\ &\quad y_2 * (y_2 * (y_1 * (y_1 * y_2)))\}, \sup\{\mu_A(x_3)|x_3 \in X, \varphi(x_3) = y_3\})\\ &= T(\varphi(\mu_A)(y_2 * (y_2 * (y_1 * (y_1 * y_2)))), \varphi(\mu_A)(y_3)).\end{aligned}$$

Thus

$$\varphi(\mu_A)(y_1 * (y_1 * y_2)) \geq T(\varphi(\mu_A)(y_2 * (y_2 * (y_1 * (y_1 * y_2)))), \varphi(\mu_A)(y_3)).$$

Also

$$\begin{aligned}&\varphi(\nu_A)(y_1 * (y_1 * y_2))\\ &= \inf\{\nu_A(x_1 * (x_1 * x_2))|x_1 * (x_1 * x_2) \in X, \varphi(x_1 * (x_1 * x_2)) = y_1 * (y_1 * y_2)\}\\ &\leq \inf\{C(\nu_A(x_2 * (x_2 * (x_1 * (x_1 * x_2)))), \nu_A(x_3))|x_i \in X, \varphi(x_i) = y_i\}\\ &= C(\inf\{\nu_A(x_2 * (x_2 * (x_1 * (x_1 * x_2))))|x_i \in X, \varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2)))) =\\ &\quad y_2 * (y_2 * (y_1 * (y_1 * y_2)))\}, \inf\{\nu_A(x_3) \mid x_3 \in X, \varphi(x_3) = y_3\})\\ &= T(\varphi(\mu_A)(y_2 * (y_2 * (y_1 * (y_1 * y_2)))), \varphi(\mu_A)(y_3)).\end{aligned}$$

Thus

$$\varphi(\nu_A)(y_1 * (y_1 * y_2)) \leq C(\varphi(\nu_A)(y_2 * (y_2 * (y_1 * (y_1 * y_2)))), \varphi(\nu_A)(y_3)).$$

Therefore $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in (T, C)IFCI(Y)$. □

Proposition 3.19. *If $B = (\mu_B, \nu_B) \in (T, C)IFCI(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFCI(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\mu_B)(0) = \mu_B(\varphi(0)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(0) = \nu_B(\varphi(0)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

Let $x_1, x_2, x_3 \in X$. Now

$$\begin{aligned}
&\varphi^{-1}(\mu_B)(x_1 * (x_1 * x_2))) \\
&= \mu_B(\varphi(x_1 * (x_1 * x_2)))) \\
&= \mu_B(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))) \\
&\geq T(\mu_B(\varphi(x_2) * (\varphi(x_2) * (\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))))), \mu_B(\varphi(x_3))) \\
&= T(\mu_B(\varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2))))), \mu_B(\varphi(x_3))) \\
&= T(\varphi^{-1}(\mu_B)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\mu_B)(x_3))
\end{aligned}$$

Then

$$\varphi^{-1}(\mu_B)(x_1 * (x_1 * x_2))) \geq T(\varphi^{-1}(\mu_B)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\mu_B)(x_3)).$$

Also

$$\begin{aligned}
&\varphi^{-1}(\nu_B)(x_1 * (x_1 * x_2))) \\
&= \nu_B(\varphi(x_1 * (x_1 * x_2)))) \\
&= \nu_B(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))) \\
&\leq C(\nu_B(\varphi(x_2) * (\varphi(x_2) * (\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))))), \nu_B(\varphi(x_3))) \\
&= C(\nu_B(\varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2))))), \nu_B(\varphi(x_3))) \\
&= C(\varphi^{-1}(\nu_B)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\nu_B)(x_3))
\end{aligned}$$

Then

$$\varphi^{-1}(\nu_B)(x_1 * (x_1 * x_2))) \leq C(\varphi^{-1}(\nu_B)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\nu_B)(x_3)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in (T, C)IFCI(X)$. □

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Department of Mathematics, Payame Noor University (PNU), P. O. Box 19395-3697, Tehran, Iran

Email address: rasulirasul@yahoo.com,

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