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On fixed point results in F-metric space with applications to neutral differential equations

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ABSTRACT. In this paper, a new concept of Ćirić-Kannan -$\alpha - \psi$-contractions in the setting of F-metric space is introduced. Using this idea, endowed with suitable hypotheses, some fixed point theorems for such mappings in F-complete F-metric space are established. As an application, novel existence conditions for a solution of nonlinear neutral differential equations are investigated to show the usability of our obtained results.

1. Introduction

Fixed point theory is a highly useful division of mathematics. Its fundamental idea hinges on conditions for existence of fixed points of maps. The Banach-Caccioppoli theorem [7] is the first limelight result in this field. It plays a significant role in several branches of sciences and has been improved by more than a few authors, for instance, see [2, 4, 8, 15, 18, 20, 21, 22, 23, 24]. Nadler [18] initiated the notions of multi-valued map and established corresponding fixed point results for such maps. Along the line, Samet etal. [19] launched the concept of $\alpha - \psi$-contractions using α-admissible maps and presented some fixed point theorems which include the contraction mapping principle due to Banach as a special case. At present, the notion of $\alpha - \psi$-contractions has been generalized by several authors, for instance, see [3, 16] and the references therein.

Not long ago, the idea of F-metric space was presented in [14] and a generalized version of the Banach fixed point result in the setting of F-metric space was

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established. Meanwhile, researchers have picked keen interests in extending results in the generalized metric space; see for instance, [1, 11]. In this paper, we define a new concept of Ćirić-Kannan $\alpha-\psi$-contraction in the framework of F-metric space. Some illustrative examples and applications to nonlinear neutral differential equations are considered to show the usability of our obtained results.

2. Preliminaries

In this section, we record specific concepts that will be needed in the sequel. Represent by Ω, the family of isotone functions ψ on $[0,\infty)$ satisfying $\sum_{j=1}^{\infty}\psi^j(t) < +\infty\ \forall\ 0<t$.

Lemma 2.1. [19] *For every self map ψ on $[0,\infty)$, the following hold: If ψ is increasing, $0<t$, $0=\lim_{j\longrightarrow\infty}\psi^j(t)$ yields $t>\psi(t)$.*

Definition 2.1. [19] Let (U,d) be an F-metric space , $\Lambda : U \longrightarrow U$ be a map. We call Λ an $\alpha-\psi$-contraction if $\alpha : U\times U \longrightarrow [0,\infty)$ and $\psi\in\Omega$ with

$$d(\Lambda\alpha,\Lambda\rho)\alpha(\alpha,\rho)\leq\psi(d(\alpha,\rho)).$$

Definition 2.2. [19] Let $\Lambda : U \longrightarrow U$ and $\alpha : U\times U \longrightarrow [0,\infty)$ be maps. We call Λ an α-admissible if

$$1\leq\alpha(\alpha,\rho)\quad\text{yields}\quad\alpha(\Lambda\alpha,\Lambda\rho)\geq 1,\quad\forall\alpha,\rho\in U.$$

Example 2.3. [19] Let $U=(0,\infty)$. Define $\Lambda : U\longrightarrow U$ and $\alpha : U\times U\longrightarrow [0,\infty)$ by

$$\Lambda\alpha=\ln(\alpha),\forall\alpha\in U$$

and

$$\alpha(\alpha,\rho)=\begin{cases}2, & \text{if } \rho\leq\alpha\\ 0, & \text{if } \ \rho>\alpha.\end{cases}$$

Then, Λ is α-admissible.

Definition 2.4. [14] Let F be the class of map $g : (0,\infty)\longrightarrow\mathbb{R}$ with:

(F_1) g is increasing, meaning

$$s<t\quad\text{implies}\quad g(s)\leq g(t).$$

(F_2) For any sequence $\{t_j\}_{j\in\mathbb{N}}$ in $(0,\infty)$,

$$\lim_{j\longrightarrow\infty}t_j=0\iff\lim_{j\longrightarrow\infty}g(t_j)=-\infty.$$

Definition 2.5. [14] Let $\emptyset\neq U$, $D : U\times U\longrightarrow[0,\infty)$ be a map. Suppose $(g,k)\in F\times[0,\infty)$ with

$(D1)$ $(\alpha,\rho)\in U\times U$, $\alpha=\rho\iff 0=D(\alpha,\rho)$;

$(D2)$ $D(\alpha,\rho)$ and $D(\rho,\alpha)$ are equal;

$(D3)$ for every $(\alpha, \rho) \in U \times U$, $n_0 \in \mathbb{N}, n_0 \geq 2$ and for every sequence $\{v_j\}_{j=1}^{n} \subset U$ with $(v_1, v_{n_0}) = (\alpha, \rho)$, we have

$$D(\alpha, \rho) > 0 \Rightarrow g(D(\alpha, \rho)) \leq g\left(\sum_{j=1}^{n_0-1} D(v_j, v_{j+1})\right) + k.$$

Then, D is called an F-metric on U and (U, D) is said to be an F-metric space.

Example 2.6. The set of all reals is an F-metric space if we take D as

$$D(\alpha, \rho) = \begin{cases} (\alpha - \rho)^2, & \text{if } [0,3] \times [0, 3\xi] \ni (\alpha, \rho), 1 \leq \xi \\ |\alpha - \rho|, & \text{if } \quad (\alpha, \rho) \notin [0, 3\xi] \times [0, 3], 1 \leq \xi \end{cases}$$

where $g(t) = \ln(t)$ and $k = \ln(3\xi)$.

Definition 2.7. [14] Let (U, D) be an F-metric space.

(i) A sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ in U is F-convergent to $\alpha \in U$ if $\{\alpha_j\}_{j\in\mathbb{N}}$ is convergent with respect to D.

(ii) $\{\alpha_j\}_{j\in\mathbb{N}}$ is F-Cauchy if

$$0 = \lim_{j,l \longrightarrow \infty} D(\alpha_j, \alpha_l).$$

(iii) (U, D) is F-complete if all F-Cauchy sequences in U is F-convergent to some point of U.

The following refinement of Banach-Caccioppoli result is due to Jlei and Samet [14].

Theorem 2.2. [14] *Let (U, D) be an F-complete F-metric space , $g : U \longrightarrow U$ be a map. Assume $\eta \in (0, 1)$ with*

$$D(g(\alpha), g(\rho)) \leq \eta D(\alpha, \rho).$$

Then g has a unique fixed point in U. Moreover, for every $\alpha_0 \in U$, the sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ defined by $\alpha_{j+1} = g(\alpha_j)$, $j \in \mathbb{N}$ is F-convergent

3. Main Results

In this section, we present the idea of Ćirić-Kannan $\alpha - \psi$-contraction in the framework of F-metric space and establish a few fixed point results under some suitable hypotheses.

Definition 3.1. Let (U, D) be an F-metric space and $\Lambda : U \longrightarrow U$ be a map. We call Λ a Kannan-type contraction, if there exists $\eta \in \left[0, \frac{1}{2}\right)$ such that

$$D(\Lambda\alpha, \Lambda\rho) \leq \eta \left[D(\alpha, \Lambda\alpha) + D(\rho, \Lambda\rho)\right], \quad \forall \alpha, \rho \in U.$$

Definition 3.2. Let (U, D) be an F-metric space $\Lambda : U \longrightarrow U$ be a map. We call Λ a *Ćirić*-Kannan $\alpha - \psi$-contraction , if there exist $\eta \in \left[0, \frac{1}{2}\right)$, $\psi \in \Omega$ and $\alpha : U \times U \longrightarrow [0, \infty)$ such that

$$\alpha(\alpha, \rho)D(\Lambda\alpha, \Lambda\rho) \leq \psi\left(A(\alpha, \rho)\right) + \eta\left[D(\alpha, \Lambda\alpha) + D(\rho, \Lambda\rho), \right] \tag{1}$$

where

$$A(\alpha, \rho) = \max\{D(\alpha, \rho), D(\alpha, \Lambda\alpha), D(\rho, \Lambda\rho)\}.$$

Theorem 3.1. *Let (U, D) be an F-complete F-metric space , $\Lambda : U \longrightarrow U$ be α-admissible and Ćirić-Kannan $\alpha - \psi$-contraction. Suppose that*

(i) *Λ is a Kannan-type contraction;*
(ii) *we have an $\alpha_0 \in U$ with $\alpha(\alpha_0, \Lambda\alpha_0) \geq 1$;*
(iii) *$\alpha(\alpha, \rho) \geq 1$, for all $\alpha, \rho \in U$;*
(iv) *every $g \in F$ satisfies $g(s) < g(t) \Longleftrightarrow s < t,\ s, t > 0$.*

Then, Λ has a unique fixed point in U.

Proof. Let $\alpha_0 \in U$ and $\alpha_j = \Lambda\alpha_{j-1}, j = 1, 2, 3, \cdots$ such that $\alpha(\alpha_0, \Lambda\alpha_0) \geq 1$. If $\alpha_{j-1} = \alpha_j$, for some $j \in \mathbb{N}$, then $\alpha^* = \alpha_j$ is a fixed point of Λ. If $\alpha_{j-1} \neq \alpha_j$, for all $j \in \mathbb{N}$. By α-admissibility of Λ, $\alpha(\alpha_0, \alpha_1) \geq 1$ implies $\alpha(\Lambda\alpha_0, \Lambda\alpha_1) = \alpha(\alpha_1, \alpha_2) \geq 1$. This process can be repeated recursively to have $\alpha(\alpha_{j-1}, \alpha_j) \geq 1,\ 1 \leq j$. By hypothesis,

$$\begin{aligned} D(\Lambda\alpha_{j-1}, \Lambda\alpha_j) &\leq D(\Lambda\alpha_{j-1}, \Lambda\alpha_j)\alpha(\alpha_{j-1}, \alpha_j) \\ &\leq \psi\left(A(\alpha_{j-1}, \alpha_j)\right) + \eta\left[D(\alpha_{j-1}, \Lambda\alpha_{j-1}) + D(\alpha_j, \Lambda\alpha_j)\right] \\ &\leq \psi\left(A(\alpha_{j-1}, \alpha_j)\right) + \eta\left[D(\alpha_{j-1}, \alpha_j) + D(\alpha_j, \alpha_{j+1})\right] \end{aligned} \tag{2}$$

.

By condition (i), for $\lambda = \frac{\eta}{1-\eta}$, we have

$$\begin{aligned} D(\alpha_{j-1}, \alpha_j) &= D(\Lambda\alpha_{j-2}, \Lambda\alpha_{j-1}) \\ &\leq \eta\left[D(\alpha_{j-2}, \Lambda\alpha_{j-2}) + D(\alpha_{j-1}, \Lambda\alpha_{j-1})\right] \\ &\leq \frac{\eta}{1-\eta}D(\alpha_{j-2}, \alpha_{j-1}) \\ &\leq \left(\frac{\eta}{1-\eta}\right)^2 D(\alpha_{j-3}, \alpha_{j-2}) \\ &\ \ \vdots \\ &\leq \lambda^j D(\alpha_0, \alpha_1). \end{aligned} \tag{3}$$

Therefore, (2) becomes

$$D(\Lambda\alpha_{j-1}, \Lambda\alpha_j) \leq \psi\left(A(\alpha_{j-1}, \alpha_j)\right) + 2\eta\lambda^j D(\alpha_0, \alpha_1). \tag{4}$$

Now, consider

$$\begin{aligned} A(\alpha_{j-1},\alpha_j) &= \max\{D(\alpha_{j-1},\alpha_j), D(\alpha_{j-1},\Lambda\alpha_{j-1}), D(\alpha_j,\Lambda\alpha_j)\} \\ &= \max\{D(\alpha_{j-1},\alpha_j), D(\alpha_{j-1},a\alpha j), D(\alpha_j,\alpha_{j+1})\} \\ &= \max\{D(\alpha_{j-1},\alpha_j), D(\alpha_j,\alpha_{j+1})\}. \end{aligned}$$

If $\max\{D(\alpha_{j-1},\alpha_j), D(\alpha_j,\alpha_{j+1})\} = D(\alpha_j,\alpha_{j+1})$, then from (4), we get

$$\begin{aligned} D(\alpha_j,\alpha_{j+1}) &\leq \psi\left(D(\alpha_j,\alpha_{j+1})\right) + 2\eta\lambda^j D(\alpha_0,\alpha_1) \\ &< D(\alpha_j,\alpha_{j+1}) + 2\eta\lambda^j D(\alpha_0,\alpha_1). \end{aligned} \tag{5}$$

Letting $\lambda \longrightarrow 0^+$ in (5), we obtain $D(\alpha_j,\alpha_{j+1}) < D(\alpha_j,\alpha_{j+1})$, is impossible. Whence,

$$\max\{D(\alpha_{j-1},\alpha_j), D(\alpha_j,\alpha_{j+1})\} = D(\alpha_{j-1},\alpha_j).$$

Therefore, (4) becomes

$$D(\alpha_j,\alpha_{j+1}) \leq \psi\left(D(\alpha_{j-1},\alpha_j)\right) + 2\eta\lambda^j D(\alpha_0,\alpha_1). \tag{6}$$

Repeating these steps recursively, we get

$$D(\alpha_j,\alpha_{j+1}) \leq \psi^j(D(\alpha_0,\alpha_1)) + 2\eta\lambda^j D(\alpha_0,\alpha_1), \forall j \in \mathbb{N}. \tag{7}$$

Let $\epsilon > 0$ and $(g,k) \in F \times \mathbb{R}^+$ with $(D3)$ satisfied. From (F_2), we get $0 < \delta$ with

$$t < \delta \quad \text{yielding} \quad g(t) < g(\epsilon) - k. \tag{8}$$

Let $j(\epsilon) \in \mathbb{N}$ such that

$$0 < \left[\sum_{j \geq j(\epsilon)} \left(\psi^j(D(\alpha_0,\alpha_1)) + 2\eta\lambda^j D(\alpha_0,\alpha_1)\right)\right] < \delta.$$

Hence, by (8) and (F_1), we have

$$\begin{aligned} g\left[\sum_{i=j}^{m-1} \psi^i(D(\alpha_0,\alpha_1)) + 2\eta D(\alpha_0,\alpha_1)\sum_{i=j}^{m-1}\lambda^i\right] &\leq g\left[\sum_{j \geq j(\epsilon)} \psi^j(D(\alpha_0,\alpha_1)) + 2\eta D(\alpha_0,\alpha_1)\sum_{j \geq j(\epsilon)}\lambda^j\right] \\ &< g(\epsilon) - k, \end{aligned} \tag{9}$$

where $m > j \geq j(\epsilon)$. Now, using $(D3)$, (6) and (9), $D(a_j,a_m) > 0$ implies

$$\begin{aligned} g(D(\alpha_j,\alpha_m)) &\leq g\left[\sum_{i=j}^{m-1} \psi^i(D(\alpha_0,\alpha_1)) + 2\eta D(\alpha_0,\alpha_1)\sum_{i=j}^{m-1}\lambda^i\right] + k \\ &\leq g\left[\sum_{j \geq j(\epsilon)} \psi^j(D(\alpha_0,\alpha_1)) + 2\eta D(\alpha_0,\alpha_1)\sum_{j \geq j(\epsilon)}\lambda^j\right] + k \\ &< g(\epsilon), \end{aligned}$$

which implies by condition (iv) that $\lim_{j,m\longrightarrow+\infty} D(\alpha_j,\alpha_m)=0$. Hence, the sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ is F-Cauchy. Completeness of (U,D) yields $\alpha^*\in U$ such that $\alpha_j\longrightarrow\alpha^*$ as $j\longrightarrow\infty$.

To see that α^* is a fixed point of Λ, assume that $\alpha^*=\Lambda\alpha^*$ so that $D(\Lambda\alpha^*,\alpha^*)>0$ with $\alpha(\alpha^*,\alpha_{j-1})\geq 1, j\in\mathbb{N}$. By $(D3)$, we get

$$\begin{aligned} g(D(\Lambda\alpha^*,\alpha^*)) &\leq g(D(\Lambda\alpha^*,\alpha_j)+D(\alpha_j,\alpha^*))+k \\ &\leq g(D(\Lambda\alpha^*,\Lambda\alpha_{j-1})+D(\alpha_j,\alpha^*))+k \\ &\leq g\left(\alpha(\alpha^*,\alpha_{j-1})D(\Lambda\alpha^*,\Lambda\alpha_{j-1})+D(\alpha_j,\alpha^*)\right)+k \\ &\leq g\left[\psi\left(A(\alpha_{j-1},\alpha^*)\right)+\eta\left(D(\alpha_{j-1},\Lambda\alpha_{j-1})+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha_j,\alpha^*)\right]+k \\ &\leq g\left[\psi\left(A(\alpha_{j-1},\alpha^*)\right)+\eta\left(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha_j,\alpha^*)\right]+k \end{aligned} \tag{10}$$

where

$$A(\alpha_{j-1},\alpha^*)=\max\{D(\alpha_{j-1},\alpha^*),D(\alpha_{j-1},\alpha_j),D(\alpha^*,\Lambda\alpha^*)\}.$$

If

$$\max\{D(\alpha_{j-1},\alpha^*),D(\alpha_{j-1},\alpha_j),D(\alpha^*,\Lambda\alpha^*)\}=D(\alpha_{j-1},\alpha^*),$$

then (10) reduces to

$$\begin{aligned} g(D(\Lambda\alpha^*,\alpha^*)) &\leq g\left[\psi(D(\alpha_{j-1},\alpha^*))+\eta\left(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha^*,\alpha_j)\right]+k \\ &< g\left[D(\alpha_{j-1},\alpha^*)+\eta\left(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha^*,\alpha_j)\right] \\ &\quad +k. \end{aligned} \tag{11}$$

By condition (iv), eq. (11) implies that

$$D(\Lambda\alpha^*,\alpha^*)<D(\alpha_{j-1},\alpha^*)+[\eta(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*))+D(\alpha^*,\alpha_j)]+k \tag{12}$$

As $j\longrightarrow\infty$ in (12), we have

$$D(\Lambda\alpha^*,\alpha^*)<\eta D(\Lambda\alpha^*,\alpha^*)+k. \tag{13}$$

Again, letting $k\longrightarrow 0^+$ in(13) , we have the contradiction $D(\Lambda\alpha^*,\alpha^*)<\eta D(\Lambda\alpha^*,\alpha^*)$, since $\eta<1$.

$$\text{If}\quad \max\{D(\alpha_{j-1},\alpha^*),D(\alpha_{j-1},\alpha_j),D(\alpha^*,\Lambda\alpha^*)\}=D(\alpha_{j-1},\alpha_j),$$

then (10) reduces to

$$\begin{aligned} g(D(\Lambda\alpha^*,\alpha^*)) &< g\left[D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\alpha_j)+\eta\left(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right)\right] \\ &\quad +k. \end{aligned}$$

On similar steps, by condition (iv),

$$D(\Lambda\alpha^*,\alpha^*)<D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\alpha_j)+\eta\left[D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right]. \tag{14}$$

Taking limit as $j\longrightarrow\infty$ in (14), yields

$$D(\Lambda\alpha^*,\alpha^*)<\eta D(\Lambda\alpha^*,\alpha^*),$$

a contradiction, given that $\eta < 1$. So, if

$$\max\{D(\alpha_{j-1}, \alpha^*), D(\alpha_{j-1}, \alpha_j), D(\alpha^*, \Lambda\alpha^*)\} = D(\alpha^*, \Lambda\alpha^*),$$

then (10) reduces to

$$g(D(\Lambda\alpha^*, \alpha^*)) < g\left[D(\alpha^*, \Lambda\alpha^*) + D(\alpha^*, \alpha_j) + \eta\left(D(\alpha_{j-1}, \alpha_j) + D(\alpha^*, \Lambda\alpha^*)\right)\right] + k.$$

By (iv),

$$D(\Lambda\alpha^*, \alpha^*) < D(\Lambda\alpha^*, \alpha^*) + D(\alpha^*, \alpha_j) + \eta\left[D(\alpha_{j-1}, \alpha_j) + D(\Lambda\alpha^*, \alpha^*)\right]. \tag{15}$$

Consequently,

$$\begin{aligned} D(\Lambda\alpha^*, \alpha^*) &< \lim_{j\longrightarrow\infty} \left(D(\Lambda\alpha^*, \alpha^*) + D(\alpha^*, \alpha_j) + \eta\left[D(\alpha_{j-1}, \alpha_j) + D(\Lambda\alpha^*, \alpha^*)\right]\right) \\ &< D(\Lambda\alpha^*, \alpha^*) + \eta D(\Lambda\alpha^*, \alpha^*) \\ &< D(\Lambda\alpha^*, \alpha^*) \quad \text{as} \quad \eta \longrightarrow 0^+, \end{aligned}$$

a contradiction. Hence $D(\alpha^*, \Lambda a^*) = 0$.

To achieve uniqueness of fixed point of Λ, suppose $\alpha^* \neq \rho^*$ are two fixed points of Λ. Then by condition (iii), we get

$$\begin{aligned} D(\alpha^*, \rho^*) &= D(\Lambda\alpha^*, \Lambda\rho^*) \\ &\leq \alpha(\alpha^*, \rho^*)D(\Lambda\alpha^*, \Lambda\rho^*) \\ &\leq \psi\left(A(\alpha^*, \rho^*)\right) + \eta\left[D(\alpha^*, \Lambda\alpha^*) + D(\rho^*, \Lambda\rho^*)\right] \\ &\leq \psi\left(\max\{D(\alpha^*, \rho^*), D(\alpha^*, \Lambda\alpha^*), D(\rho^*, \Lambda\rho^*)\}\right) + \eta(0) \\ &\leq \psi(D(\alpha^*, \rho^*)) < D(\alpha^*, \rho^*), \end{aligned}$$

a contradiction. It follows that $\alpha^* = \rho^*$. □

Example 3.3. Let $U = \mathbb{R}$ and define $D : U \times U \longrightarrow \mathbb{R}$ by

$$D(\alpha, \rho) = |\alpha - \rho|.$$

Note that D is an F-metric with $f(t) = \ln(t)$ and $k = 0$. Define $\Lambda : U \longrightarrow U$ by

$$\Lambda\alpha = \frac{\alpha + 5}{2}$$

and

$$\alpha(\alpha, \rho) = \begin{cases} \frac{4}{3}, & \text{if } \alpha, \rho \in [0, \infty) \\ 0, & \text{otherwise.} \end{cases}$$

Choose $\psi(t) = \frac{2t}{3}$ for $t \geq 0$. Clearly, Λ is an Kannan-type contraction and Ćirić-Kannan $\alpha - \psi$-contractive. In fact, for all $\alpha, \rho \in U$,

$$\begin{aligned}
\alpha(\alpha,\rho)D(\Lambda\alpha,\Lambda\rho) &\leq \frac{2}{3}|\alpha-\rho| \\
&\leq \frac{2}{3}\left(A(\alpha,\rho)\right) \\
&\leq \psi(A(\alpha,\rho)) + \eta\left[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)\right].
\end{aligned}$$

Moreover, for $\alpha_0 = 5$, we have $\alpha(\alpha_0, \Lambda\alpha_0) \geq 1$. By definition of Λ and α, it is apparent that for all $\alpha, \rho \in U$, $\alpha(\alpha,\rho) \geq 1$ implies $\alpha(\Lambda\alpha, \Lambda\rho) \geq 1$. Hence, Λ is α-admissible. Consequently, all the hypotheses of Theorem 3.1 hold. It can be seen that $\alpha^* = 5$ is a fixed point of Λ.

By equating $A(\alpha,\rho)$ to $D(\alpha,\rho)$ in Theorem 3.1, yields:

Corollary 3.2. *Let (U, D) be an F-complete F-metric space and $\Lambda : U \longrightarrow U$ be α-admissible with*

$$\alpha(\alpha,\rho)D(\Lambda\alpha,\Lambda\rho) \leq \psi(D(\alpha,\rho)).$$

Moreover, if there exists $\alpha_0 \in U$ such that $\alpha(\alpha_0, \Lambda\alpha_0) \geq 1$, then Λ has a fixed point in U.

Remark 3.4. Set $\eta = 0$, $\alpha(\alpha,\rho) = 1$ and $\psi(t) = \theta t, \quad \theta \in (0,1)$ in Corollary 3.2, gives [**14**, Theorem 1.1].

Further, in Theorem 3.1, several corollaries can be obtained if we set $\alpha(\alpha,\rho) = 1$ and the *Ćirić*-Kannan $\alpha - \psi$-contractive condition by

(i) $D(\Lambda\alpha,\Lambda\rho) \leq \psi(D(\alpha,\rho)) + \eta[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)]$
(ii) $D(\Lambda\alpha,\Lambda\rho) \leq \psi\left(\max\{D(\alpha,\Lambda\alpha), D(\rho,\Lambda\rho)\}\right) + \eta[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)]$
(iii) $D(\Lambda\alpha,\Lambda\rho) \leq \psi\left(\frac{D(\alpha,\Lambda\alpha)+D(\rho,\Lambda\rho)}{2}\right) + \eta[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)]$.

By adding continuity condition to the axioms of Theorem 3.1, yields the following result.

Theorem 3.3. *Let (U, D) be an F-complete F-metric space , $\Lambda : U \longrightarrow U$ be α-admissible and Ćirić-Kannan $\alpha - \psi$-contraction. Suppose:*

(i) *Λ is a Kannan-type contraction;*
(ii) *we have an $\alpha_0 \in U$ such that $\alpha(\alpha_0, \Lambda\alpha_0) \geq 1$;*
(iii) *every $g \in F$ satisfies $g(s) < g(t) \Longleftrightarrow s < t$;*
(iv) *the function $g \in F$ is continuous and every $\psi \in \Omega$ is taken to be continuous and satisfies $g(t) > g\left(\psi(t) + \eta t\right) + k$, $0 < t$, $\eta \in [0, \frac{1}{2})$ and $k \geq 0$.*

Then Λ has a unique fixed point.

Proof. In line with arguments in the proof of Theorem 3.1, we have that $\{\alpha_j\}_{j\in\mathbb{N}}$ is F-Cauchy, hence by completeness of (U,D), there exists $\alpha^* \in U$ with

$$\lim_{j\longrightarrow\infty} D(\alpha_j,\alpha^*) = 0. \tag{16}$$

To show that $\alpha^* = \Lambda\alpha^*$, assume that $D(\alpha^*,\Lambda\alpha^*) = t > 0$ with $\alpha(\alpha^*,\alpha_{j-1}) \geq 1, j \in \mathbb{N}$. By $(D3)$ and the fact that Λ is a C$\acute{i}ric$-Kannan $\alpha-\psi$-contraction, we get

$$\begin{aligned} g(D(\Lambda\alpha^*,\alpha^*)) &\leq g(D(\Lambda\alpha^*,\alpha_j)+D(\alpha_j,\alpha^*))+k \\ &\leq g(D(\Lambda\alpha^*,\Lambda\alpha_{j-1})+D(\alpha_j,\alpha^*))+k \\ &\leq g\left(\alpha(\alpha^*,\alpha_{j-1})D(\Lambda\alpha^*,\Lambda\alpha_{j-1})+D(\alpha_j,\alpha^*)\right)+k \\ &\leq g\left[\psi\left(A(\alpha_{j-1},\alpha^*)\right)+\eta\left(D(\alpha_{j-1},\Lambda\alpha_{j-1})+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha_j,\alpha^*)\right]+k \\ &\leq g\left[\psi\left(A(\alpha_{j-1},\alpha^*)\right)+\eta\left(D(\alpha_{j-1},\alpha_j)+D(\alpha^*,\Lambda\alpha^*)\right)+D(\alpha_j,\alpha^*)\right]+k \end{aligned} \tag{17}$$

Taking limit in (17) as $j \longrightarrow \infty$ and using (16) together with continuity of g and ψ, we obtain

$$g(D(\Lambda\alpha^*,\alpha^*)) \leq g\left(\psi(D(\alpha^*,\Lambda\alpha^*))+\eta D(\alpha^*,\Lambda\alpha^*)\right)+k,$$

a contradiction in line with the hypothesis (iv). Consequently, $D(\alpha^*,\Lambda\alpha^*) = 0$. Whence, α^* is a fixed point of Λ. □

Coupled fixed points

We recall some relevant concepts and connect our main result with the ideas in [6].

Definition 3.5. [6] Let $G : U \times U \longrightarrow U$ be a map. We call $(\alpha,\rho) \in U \times U$ a coupled fixed point of G if

$$G(\alpha,\rho) = \alpha \quad \text{and} \quad G(\rho,\alpha) = \rho.$$

Lemma 3.4. *Let (U,D) be an F-metric space and $G : U \times U \longrightarrow U$ be a map. Define $\Lambda : U \times U \longrightarrow U \times U$ by*

$$\Lambda(\alpha,\rho) = \left(G(\alpha,\rho),G(\rho,\alpha)\right), \forall(\alpha,\rho) \in U \times U. \tag{18}$$

Then (α,ρ) is mapped by G to itself $\Longleftrightarrow$ (α,ρ) is mapped by Λ to itself.

Proof. Let $\Lambda(\alpha,\rho) = (\alpha,\rho)$. Whence,

$$(\alpha,\rho) = \Lambda(\alpha,\rho) = (G(\alpha,\rho),G(\rho,\alpha))$$

implies

$$\alpha = G(\alpha,\rho) \quad \text{and} \quad \rho = G(\rho,\alpha).$$

Again, suppose (α,ρ) is a coupled fixed point of G;

$$\alpha = G(\alpha,\rho) \quad \text{and} \quad \rho = G(\rho,\alpha).$$

Now,

$$\Lambda(\alpha,\rho) = (G(\alpha,\rho), G(\rho,\alpha) = (\alpha,\rho).)$$

Hence, (α,ρ) is a fixed point of Λ. □

Theorem 3.5. *Let (U, D) be an F-complete F-metric space and $G : U\times U \longrightarrow U$ a map. Suppose that there exist $\psi \in \Omega$ and a map $\alpha : U^2 \times U^2 \longrightarrow [0,\infty)$ such that the following conditions are satisfied:*

(i) *for all $(\alpha,\rho),(\beta,\delta) \in U \times U$,*

$$\alpha\left((\alpha,\rho),(\beta,\delta)\right) \geq 1 \Rightarrow \alpha\left((G(\alpha,\rho)), G(\rho,\alpha)),(G(\beta,\delta), G(\delta,\beta))\right) \geq 1$$

(ii) *there exists $(\alpha_0,\rho_0) \in U \times U$ such that*

$$\alpha\left((\alpha_0,\rho_0),(G(\alpha_0,\rho_0), G(\rho_0,\alpha_0))\right) \geq 1$$

and

$$\alpha\left(G(\rho_0,\alpha_0), G(\alpha_0,\rho_0)\right) \geq 1;$$

(iii) *for all $(\alpha,\rho),(\beta,\delta) \in U^2 \times U^2$, there exists $\eta \in [0,\frac{1}{2})$ such that*

$$\begin{aligned}\alpha\left((\alpha,\rho),(\beta,\delta)D(G(\alpha,\rho), G(\beta,\delta))\right) \;\; &\leq \;\; \frac{1}{2}\psi\left(A(\alpha,\beta,\beta,\delta)\right) \\ &+\frac{\eta}{2}\left[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)\right],\end{aligned}$$

where

$$\begin{aligned}A(\alpha,\rho,\beta,\delta) = \max\{&D(\alpha,\beta) + D(\rho,\delta), D(\alpha, G(\alpha,\rho)) + D(\rho, G(\rho,\alpha)),\\ &D(\beta, G(\beta,\delta)) + D(\delta, G(\delta,\beta))\}.\end{aligned}$$

Then G has a coupled fixed point.

Proof. First, we will take the situation to an F-complete space (Y, D_1), with $U \times U = Y$ and

$$D_1\left((\alpha,\rho),(\beta,\delta)\right) = D(\alpha,\beta) + D(\rho,\delta), \forall(\alpha,\rho),(\beta,\delta) \in U^2 \times U^2.$$

From (iii),

$$\begin{aligned}\alpha\left((\alpha,\rho),(\beta,\delta)\right) D\left(G(\alpha,\rho), G(\beta,\delta)\right) \;\; &\leq \;\; \frac{1}{2}\psi\left(A(\alpha,\rho,\beta,\delta)\right) \\ &+\frac{\eta}{2}\left[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)\right] \qquad (19)\end{aligned}$$

and

$$\begin{aligned}\alpha\left((\delta,\beta),(\rho,\alpha)\right) D\left(G(\delta,\beta), G(\rho,\beta)\right) \;\; &\leq \;\; \frac{1}{2}\psi\left(A(\alpha,\rho,\beta,\delta)\right) \qquad (20)\\ &+\frac{\eta}{2}\left[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)\right]\end{aligned}$$

Adding (19) and (20), we get

$$\pi(r,u)D_1(\Lambda r,\Lambda u) \leq \psi(r,u) + \eta\left[D(\alpha,\Lambda\alpha) + D(\rho,\Lambda\rho)\right] \qquad (21)$$

for $r = (\alpha, \rho), u = (\beta, \delta)$ in Y, with $\pi : Y \times Y \longrightarrow [0, \infty)$ is taken as

$$\pi(r, u) = \min\{\alpha\left((\alpha, \rho), (\beta, \delta)\right), \alpha\left((\delta, \beta), (\rho, \alpha)\right)\}$$

and $\Lambda : Y \longrightarrow Y$ is defined by (18). It follows that Λ is a Ćirić-Kannan $\pi - \psi$ contractive mapping.

Let $r = (r_1, r_2), u = (u_1, u_2) \in Y$ such that $1 \leq \pi(u, r)$. Whence, (i) yields $1 \leq \pi(\Lambda r, \Lambda u)$; meaning Λ is π-admissible. Further, by (ii), we get $(\alpha_0, \rho_0) \in Y$ with

$$\pi\left((\alpha_0, \rho_0), \Lambda(\alpha_0, \rho_0) \geq 1\right).$$

Therefore, all the axioms of Theorem 3.1 hold. Consequently, we infer the conclusion. □

Example 3.6. Let $U = [0, \infty)$ be equipped with F-metric D given by

$$D(\alpha, \rho) = \begin{cases} (\alpha - \rho)^2, & \text{if } (\alpha, \rho) \in [0,3] \times [0,3] \\ |\alpha - \rho|, & \text{if } \ (\alpha, \rho) \notin [0,3] \times [0,3] \end{cases}$$

where $g(t) = \ln(t)$, $k = \ln(3)$. Then (U, D) is an F-complete F-metric space. Define G as

$$G(\alpha, \rho) = \begin{cases} \frac{1}{6}(\alpha - \rho), & \text{if } \alpha \geq \rho \\ 0, & \text{if } \ \alpha < \rho \end{cases}$$

and $\alpha : U^2 \times U^2 \longrightarrow [0, \infty)$ by

$$\alpha\left((\alpha, \rho), (\beta, \delta)\right) = \begin{cases} 2, & \text{if } \alpha \geq \rho, \beta \geq \delta \\ 0, & \text{if } \ \alpha < \rho, \beta < \delta \end{cases}$$

and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Now, for $(\alpha, \rho), (\beta, \delta) \in U^2 \times U^2$,
Case (i) If $(\alpha, \rho), (\beta, \delta) \in [0,3] \times [0,3]$, then

$$\begin{aligned} \alpha\left((\alpha, \rho), (\beta, \delta)\right) D(G(\alpha, \rho), G(\beta, \delta)) &\leq \frac{1}{18}\left[D(\alpha, \beta) + D(\rho, \delta)\right] \\ &\leq \frac{1}{18} A\left(\alpha, \rho, \beta, \delta\right) \\ &\leq \frac{1}{2}\psi\left(A(\alpha, \rho, \beta, \delta)\right) + \frac{\eta}{2}\left[D(\alpha, \Lambda\alpha) + D(\rho, \Lambda\rho)\right]. \end{aligned}$$

Case(ii) If $(\alpha, \rho), (\beta, \delta) \notin [0,3] \times [0,3]$, then

$$\begin{aligned} \alpha\left((\alpha, \rho), (\beta, \delta)\right) D(G(\alpha, \rho), G(\beta, \delta)) &\leq \frac{1}{6}\left[D(\alpha, \beta) + D(\rho, \delta)\right] \\ &\leq \frac{1}{2}\psi\left(A(\alpha, \rho, \beta, \delta)\right) + \frac{\eta}{2}\left[D(\alpha, \Lambda\alpha) + D(\rho, \Lambda\rho)\right]. \end{aligned}$$

By taking $(\alpha_0, \rho_0) = (1, 1)$, conditions (ii) and (iii) hold. Hence, all the axioms of Theorem 3.5 hold. Whence, $(0, 0)$ is a coupled fixed point of G.

Corollary 3.6. *Let (U, D) be an F-complete F-metric space and $G : U \times U \longrightarrow U$ a map. Suppose that there exist $\psi \in \Omega$ and a map $\alpha : U^2 \times U^2 \longrightarrow [0, \infty)$ satisfying the following conditions:*

(i) *for all $(\alpha, \rho), (\beta, \delta) \in U^2 \times U^2$,*

$$\alpha\left((\alpha, \rho), (\beta, \delta)\right) \geq 1 \Rightarrow \alpha\left((G(\alpha, \rho), G(\rho, \alpha)), (G(\beta, \delta), G(\delta, \beta))\right) \geq 1;$$

(ii) *there exists $(\alpha_0, \rho_0) \in U \times U$ with*

$$\alpha\left((\alpha_0, \rho_0), (G(\alpha_0, \rho_0), G(\rho_0, \alpha_0))\right) \geq 1$$

and

$$\alpha\left(G(\rho_0, \alpha_0), G(\alpha_0, \rho_0)\right) \geq 1;$$

(iii) *for all $(\alpha, \rho), (\beta, \delta) \in U^2 \times U^2$,*

$$\alpha\left((\alpha, \rho), (\beta, \delta)\right) D(G(\alpha, \rho), G(\beta, \delta)) \quad \leq \quad \frac{1}{2}\psi\left(A(\alpha, \rho, \beta, \delta)\right),$$

where

$$\max\{D(\alpha, \beta) + D(\rho, \delta), D(\alpha, G(\alpha, \rho)) + D(\rho, G(\rho, \alpha)), D(\beta, G(\beta, \delta)) + D(\delta, G(\delta, \beta))\}$$
$$= A(\alpha, \rho, \beta, \delta).$$

Then G has a coupled fixed point.

Proof. Setting $\eta = 0$ in Theorem 3.5 completes the proof. □

4. Applications

In this section, using Theorem 3.1, we provide existence conditions for a solution of a nonlinear neutral differential equation given as

$$\alpha'(t) = -w(t)\alpha(t) + v(t)h\left(\alpha(t) - r(t)\right) + z(t)\alpha'(t - r(t)) \tag{22}$$

where $r(t), w(t)$ are continuous, $z(t)$ is continuously differentiable and $r(t) > 0$ for all $t \in \mathbb{R}$ is twice continuously differentiable.

Lemma 4.1. [9] *Let $1 \leq r'(t)$, $t \in \mathbb{R}$. The function $\alpha(t)$ is a solution of (22) iff*

$$\begin{aligned}
\alpha(t) \quad &= \quad \left(\alpha(0) - \frac{z(0)}{1 - r'(0)}\alpha(-r(0))\right) e^{-\int_0^t w(s)ds} \\
&\quad + \frac{z(t)}{1 - r'(t)}\alpha(t - r(t)) \\
&\quad - \int_0^t \left(g(u)\alpha(u - r(u)) - v(u)h(\alpha(u - r(u)))\right) e^{-\int_u^t w(s)ds} du,
\end{aligned}$$

where

$$g(u) = \frac{r''(u)z(u) + (z'(u) + z(u)w(u))(1 - r'(u))}{(1 - r'(u))^2} \tag{23}$$

Suppose $\gamma : (-\infty, 0] \longrightarrow \mathbb{R}$ is a continuous and bounded function. We call $\alpha(t) := \alpha(t, 0, \gamma)$ is a solution of (22) if $\alpha(t) = \gamma(t)$,$0 \geq t$ and (22) holds for $t \geq 0$. Take $U = C(\mathbb{R}, \mathbb{R})$ to be the space of continuous functions on $\mathbb{R}$. Define A_γ as

$$A_\gamma = \{\varphi : \mathbb{R} \longrightarrow \mathbb{R}, \varphi(t) = \gamma(t), \quad \text{if} \quad t \leq 0, \varphi(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \varphi \in U\}.$$

Endowed with Chebyshev norm $\| \cdot \|$, A_γ becomes a Banach space. Define $\rho : A_\gamma \times A_\gamma \longrightarrow \mathbb{R}$ by

$$\rho(t, t^*) = \|t - t^*\| = \sup_{\alpha \in I} |t(\alpha) - t(\alpha^*)| \quad t, t^* \in A_\gamma.$$

Theorem 4.2. *Let the function $\Lambda : A_\gamma \longrightarrow A_\gamma$ be defined as*

$$\begin{aligned}(\Lambda\varphi)(t) \;&=\; \left(\varphi(0) - \frac{z(0)}{1 - r'(0)}\varphi(-r(0))\right) e^{-\int_0^t w(s)ds} \\ &\quad + \frac{z(t)}{1 - r'(t)}\varphi(t - r(t)) \\ &\quad - \int_0^t \left(g(u)\varphi(u - r(u)) - v(u)h(\varphi(u) - r(u))\right) e^{-\int_u^t w(s)ds} du,\end{aligned} \tag{24}$$

Suppose that we have:

(i) $0 \leq \lambda$, $\eta \in \left[0, \frac{1}{2}\right)$, $\psi \in \Omega$ *with*

$$\begin{aligned}&\int_0^t |g(u) \left(\varphi(u - r(u)) - \phi(u - r(u))\right)| e^{-\int_u^t w(s)ds} du \\ \leq\; & \frac{\lambda}{2}\psi \left(\max\{\|\varphi - \phi\|, \|\varphi - \Lambda\varphi\|, \|\phi - \Lambda\phi\|\}\right) \\ &+ \frac{\eta}{2} \left[\|\varphi - \Lambda\varphi\| + \|\phi - \Lambda\phi\|\right]\end{aligned}$$

and

$$\begin{aligned}&\int_0^t |v(u)h(\varphi(u) - r(u)) - h(\phi(u - r(u)))| e^{-\int_u^t w(s)ds} du \\ \leq\; & \frac{\lambda}{2}\psi \left(\max\{\|\varphi - \phi\|, \|\varphi - \Lambda\varphi\|, \|\phi - \Lambda\phi\|\}\right) \\ &+ \frac{\eta}{2} \left[\|\varphi - \Lambda\varphi\| + \|\phi - \Lambda\phi\|\right]\end{aligned}$$

for all $\varphi, \phi \in A_\gamma$.

(ii) $\left|\frac{z(t)}{1 - r'(t)}\right| + \lambda \leq 1, \quad t \geq 0.$

Then Λ has a fixed point in A_γ.

Proof. Define $\alpha : U \times U \longrightarrow \mathbb{R}$ by

$$\alpha(\alpha, \rho) = \begin{cases} 1, & \text{if } \alpha, \rho \in A_\gamma \\ 0, & \text{otherwise.} \end{cases}$$

For $\alpha,\rho\in A_\gamma$ such that $\alpha(\alpha,\rho)\geq 1$, we have $\alpha(\Lambda\alpha,\Lambda\rho)\geq 1$; hence Λ is α-admissible. Let $\varphi,\phi\in A_\gamma$. Then consider

$$\begin{aligned}
|(\Lambda\varphi)(t)-(\Lambda\phi)(t)| &\leq \left|\frac{z(t)}{1-r'(t)}\right| \|\varphi-\phi\| \\
&\quad + \int_0^t |g(u)(\varphi(u-r(u)))-\phi(u-r(u))|\, e^{-\int_u^t w(s)ds} du \\
&\quad + \int_0^t |v(u)h\left(\varphi(u-r(u))\right)-h\left(\phi(u-r(u))\right)|\, e^{-\int_u^t w(s)ds} du \\
&\leq \left|\frac{z(t)}{1-r'(t)}\right| \psi\left(\max\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\}\right) \\
&\quad + \frac{\lambda}{2}\psi\left(\max\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\}\right) \\
&\quad + \frac{\eta}{2}\left[\|\varphi-\Lambda\varphi\|+\|\phi-\Lambda\phi\|\right] \\
&\quad + \frac{\lambda}{2}\psi\left(\max\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\}\right) \\
&\quad + \frac{\eta}{2}\left[\|\varphi-\Lambda\varphi\|+\|\phi-\Lambda\phi\|\right] \\
&\leq \left(\left|\frac{z(t)}{1-r'(t)}\right|+\lambda\right)\psi\left(\max\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\}\right) \\
&\quad + \eta\left[\|\varphi-\Lambda\varphi\|+\|\phi-\Lambda\phi\|\right] \\
&\leq \psi\left(\max\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\}\right) \\
&\quad + \eta\left[\|\varphi-\Lambda\varphi\|+\|\phi-\Lambda\phi\|\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha(\varphi,\phi)\,\|(\Lambda\varphi)(t)-(\Lambda\phi)(t)\| &= \rho(\Lambda\varphi,\Lambda\phi) \\
&\leq \psi\left(A(\varphi,\phi)\right)+\eta\left[\rho(\varphi,\Lambda\varphi)+\rho(\phi,\Lambda\phi)\right],
\end{aligned}$$

where

$$A(\varphi,\phi)=\max\left\{\|\varphi-\phi\|,\|\varphi-\Lambda\varphi\|,\|\phi-\Lambda\phi\|\right\}.$$

This implies that Λ is a *Ćirić*-Kannan $\alpha-\psi$-contraction. Moreover, $\alpha(\alpha,\rho)=1$ for all fixed points $\alpha,\rho\in A_\gamma$ of Λ. Consequently, all the axioms of Theorem 3.1 hold. Whence, Λ has a unique fixed point . It follows that (22) has a solution. □

Conclusion

In this note, an idea of *Ćirić*-Kannan $\alpha-\psi$-contraction is introduced. This concept is used to obtain some new fixed point results in the context of F-metric space. The Presented notions herein improved the work of [**14**, Th.1.1] and the main theorem of [**19**]. Knowing that mapping is a basic concept in mathematics and allied areas of sciences, the authors are hopeful that this paper will contribute not only to the area of metric fixed point but to the entire realm of science and engineering where mappings have applications.

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