Mathematical Analysis and its Contemporary Applications
Volume 4, Issue 3, 2022, 63–83
doi: 10.30495/maca.2022.1949822.1046
ISSN 2716-9898

A novel coupled fixed point results pertinent to $\mathscr{A}_b$-metric spaces with application to integral equations

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Abstract. We prove the existence and uniqueness of coupled common fixed point results in partially ordered $\mathscr{A}_b$-metric spaces in the present study. In addition to the acquired results, examples and applications are provided.

1. Introduction

Adopting fixed point strategy to solve fractional differentials, integral equations, and boundary problems is often a nightmare. It is because we must deal with not just a high number of contradictions, but also their non-static nature. In recent times, scholars have become interested in fixed point theory (simply, FPT), which was first proposed by a Polish mathematician. Real-world challenges include ground water flow problems, fractional advection dispersion equations, spatial frequency diffusion equation models, lumped parameter models, linguistic factors, and metaphysics. Quasi geometry, crystallisation theory, and coding analytics all benefit from FPT. It has a wide range of applications in medical innovation, biomechanics, and algorithms.

The idea of metric spaces is a broad concept that underpins many fields of mathematics, including topological structures, complex concepts, etc. Many fascinating generalisations (or adaptations) of the metric space idea have proposed over the years (see for example 2.4-2.6). The concept of a b-metric was first suggested by I.A.Bakhtin [4]. M.Abbas et al. [2] recently developed the idea of n-tuple metric space and investigated its topological features. As an extensive sense of n-tuple metric space, M.Ughade et al. [12] proposed the concept of $\mathscr{A}_b$-metric spaces. In

2020 *Mathematics Subject Classification.* Primary: 34A12 ; Secondary: 54H25.

Key words and phrases. Mixed weakly monotone property, $\mathscr{A}$-metric space, $\mathscr{A}_b$-metric with index n, $\mathscr{A}_b$-metric space, Integral equation.

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partially ordered $\mathscr{A}_b$-metric spaces, N.Mlaiki et al. [10] obtained unique coupled common fixed point theorems.

We prove various unique coupled common fixed point theorems in partially ordered $\mathscr{A}_b$-metric space using the preceding facts, as well as we present examples and applications to demonstrate our results.

We'll need some basic definitions, results, and examples from the literature before we can prove the main results.

2. Preliminaries

Definition 2.1. ([2]). Let $\mathscr{M}$ be a non empty set and $\gamma(\geq 2)$ be a positive integer. A function $\mathscr{A} : \mathscr{M}^\gamma \to [0, \infty)$ is called an $\mathscr{A}$-metric on $\mathscr{M}$, if for any α_i, a $\in \mathscr{M}$; $i = 1, 2, \ldots, \gamma$, the following conditions hold.
$(i)\mathscr{A}(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) \geq 0$,
$(ii)\mathscr{A}(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) = 0$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_{\gamma-1} = \alpha_\gamma$,

$$\begin{aligned}(iii)\mathscr{A}(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) \leq [&\mathscr{A}(\alpha_1, \alpha_1, \ldots, \alpha_{1_{(\gamma-1)}}, a) + \mathscr{A}(\alpha_2, \alpha_2, \ldots, \alpha_{2_{(\gamma-1)}}, a)\\ &+ \cdots + \mathscr{A}(\alpha_{\gamma-1}, \alpha_{\gamma-1}, \ldots, \alpha_{\gamma-1_{(\gamma-1)}}, a)\\ &+ \mathscr{A}(\alpha_\gamma, \alpha_\gamma, \ldots, \alpha_{\gamma_{(\gamma-1)}}, a)].\end{aligned}$$

The pair $(\mathscr{M}, \mathscr{A})$ is called an $\mathscr{A}$-metric space.

Definition 2.2. ([6]). Let $\mathscr{M}$ be a non empty set. $\mathscr{A}_b$-metric on $\mathscr{M}$ is a function $d : \mathscr{M}^2 \to [0, \infty)$ such that the following conditions hold for all $\alpha, \beta, \mu \in \mathscr{M}$.
(i) $d(\alpha, \beta) = 0 \iff \alpha = \beta$,
(ii) $d(\alpha, \beta) = d(\beta, \alpha)$,
(iii) there exists $s \geq 1$, such that $d(\alpha, \mu) \leq s[d(\alpha, \beta) + d(\beta, \mu)]$.
The pair $(\mathscr{M}, d)$ is called a b-metric space.

Definition 2.3. ([12]). Let $\mathscr{M}$ be a non empty set and $n \geq 2$. Suppose $b \geq 1$ is a real number. A function $\mathscr{A}_b : \mathscr{M}^n \to [0, \infty)$ is called an $\mathscr{A}_b$ -metric on $\mathscr{M}$, if for any $\alpha_i, a \in \mathscr{M}$, $i = 1, 2, \ldots, \gamma$, the following conditions hold.
(i) $\mathscr{A}_b(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) \geq 0$,
(ii) $\mathscr{A}_b(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) = 0$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_{\gamma-1} = \alpha_\gamma$,

$$\begin{aligned}(iii)\ \mathscr{A}_b(\alpha_1, \alpha_2, \ldots, \alpha_{\gamma-1}, \alpha_\gamma) \leq b[&\mathscr{A}_b(\alpha_1, \alpha_1, \ldots, \alpha_{1_{(\gamma-1)}}, a)\\ &+ \mathscr{A}_b(\alpha_2, \alpha_2, \ldots, \alpha_{2_{(\gamma-1)}}, a)\\ &+ \cdots + \mathscr{A}_b(\alpha_{\gamma-1}, \alpha_{\gamma-1}, \ldots, \alpha_{\gamma-1_{(\gamma-1)}}, a)\\ &+ \mathscr{A}_b(\alpha_\gamma, \alpha_\gamma, \ldots, \alpha_{\gamma_{(\gamma-1)}}, a)].\end{aligned}$$

The pair $(\mathscr{M}, \mathscr{A}_b)$ is called an $\mathscr{A}_b$-metric space.

The metric space $\mathscr{A}_b$ seems to be more versatile than the metric space $\mathscr{A}$. Likewise, $mathscrA_b$-metric space with $b = 1$ is an unique circumstance of $\mathscr{A}_b$-metric space. We observe because b-metric space and S_b-metric space with $n = 2$ and 3, respectively, are particular case of $\mathscr{A}_b$-metric space. Usual metric space and S-metric space are indeed particular variants of $\mathscr{A}_b$-metric space, with b=1 and $\gamma = 2$ and 3 correspondingly.

In this scenario, γ is called the index of $\mathscr{A}_b$.

Example 2.4. ([12]). Let $\mathscr{M} = [0, \infty)$ and $\gamma \geq 2$. Define $\mathscr{A}_b : \mathscr{M}^\gamma \to [0, \infty)$ by $\mathscr{A}_b(\alpha_1, \alpha_2, \dots, \alpha_{\gamma-1}, \alpha_\gamma) = \sum_{i=1}^{\gamma} \sum_{i<j} |\alpha_i - \alpha_j|^2$, for all $\alpha_i \in \mathscr{M}$, $i = 1, 2, \dots, \gamma$. Then $(\mathscr{M}, \mathscr{A}_b)$ is an $\mathscr{A}_b$-metric space with $b = 2$.

Example 2.5. ([12]). Let $\mathscr{M} = \mathbb{R}$ and $\gamma \geq 2$. Define $\mathscr{A}_b : \mathscr{M}^\gamma \to [0, \infty)$ by

$$\begin{aligned}\mathscr{A}_b(\alpha_1, \alpha_2, \dots, \alpha_{\gamma-1}, \alpha_\gamma) = \sum_{i=2}^{\gamma} |\alpha_i - (\gamma - 1)\alpha_1|^2 + \sum_{i=3}^{\gamma} |\alpha_i - (\gamma - 2)\alpha_2|^2 + \\ \dots + \sum_{i=\gamma-2}^{n} |\alpha_i - 3\alpha_{\gamma-3}|^2 + \sum_{i=\gamma-1}^{\gamma} |\alpha_i - 2\alpha_{\gamma-2}|^2 \\ + |\alpha_\gamma - \alpha_{\gamma-1}|^2,\end{aligned}$$

for all $\alpha_i \in \mathscr{M}$, $i = 1, 2, \dots, \gamma$. Then $(\mathscr{M}, \mathscr{A}_b)$ is an $\mathscr{A}_b$-metric space with $b = 2$.

Example 2.6. ([10]). Let $\mathscr{M} = [0, \infty)$ and $\gamma \geq 2$. Define $\mathscr{A}_b : \mathscr{M}^\gamma \to [0, \infty)$ by $\mathscr{A}_b(\alpha_1, \alpha_2, \dots, \alpha_{\gamma-1}, \alpha_\gamma) = |\alpha_1 - max(\alpha_2, \alpha_3, \dots, \alpha_\gamma)|^2$, for all $\alpha_i \in \mathscr{M}$, $i = 1, 2, \dots, \gamma$. Then $(\mathscr{M}, \mathscr{A}_b)$ is an $\mathscr{A}_b$-metric space on $\mathscr{M}$ with $b = 2$, and it is not difficult to see that $(\mathscr{M}, \mathscr{A}_b)$ is not an A-metric space on $\mathscr{M}$.

Lemma 2.1. *([12]). Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$ metric space, so that $A : \mathscr{M}^\gamma \to [0, \infty)$ for some $\gamma \geq 2$. Then $A(\underbrace{\alpha, \alpha, \dots, \alpha}_{(\gamma-1)times}, \beta) \leq b\mathscr{A}(\underbrace{\beta, \beta, \dots, \beta}_{(\gamma-1)times}, \alpha)$, for all $\alpha, \beta \in \mathscr{M}$*

Lemma 2.2. *([12]). Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$ metric space, so that $\mathscr{A} : \mathscr{M}^\gamma \to [0, \infty)$ for some $\gamma \geq 2$.
Then $\mathscr{A}(\underbrace{\alpha, \alpha, \dots, \alpha}_{(\gamma-1)times}, \mu) \leq (\gamma - 1)b\mathscr{A}(\underbrace{\alpha, \alpha, \dots, \alpha}_{(\gamma-1)times}, \beta) + b^2\mathscr{A}(\underbrace{\beta, \beta, \dots, \beta}_{(\gamma-1)times}, \mu)$, for all $\alpha, \beta, \mu \in \mathscr{M}$.*

Lemma 2.3. *([12]). Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$-metric space. Then $(\mathscr{M}^2, D_A)$ is $\mathscr{A}_b$-metric space on $\mathscr{M} \times \mathscr{M}$ with D_A defined by
$D_A((\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_\gamma, \beta_\gamma)) = \mathscr{A}(\alpha_1, \alpha_2, \dots, \alpha_\gamma) + \mathscr{A}(\beta_1, \beta_2, \dots, \beta_\gamma)$, for all $\alpha_i, \beta_i \in \mathscr{M}$, $i, j = 1, 2, \dots, \gamma$.*

Definition 2.7. . Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$-metric space. A sequence $\{\alpha_\theta\}$ in $\mathscr{M}$ is said to converge to a point $\alpha \in \mathscr{M}$, if $\mathscr{A}(\underbrace{\alpha_\theta, \alpha_\theta, \dots, \alpha_\theta}_{(n-1)times}, \alpha) \to 0$ as $\theta \to \infty$.

That is, to each $\varepsilon \geq 0$ there exist $N \in \mathbb{N}$ such that for all $k \geq N$, we have $\mathscr{A}(\underbrace{\alpha_\theta, \alpha_\theta, \ldots, \alpha_\theta}_{(n-1)times}, \alpha) \leq \varepsilon$ and we write $\lim_{\theta\to\infty} \alpha_\theta = \alpha$.

Note: α is called the limit of the sequence $\{\alpha_\theta\}$

Lemma 2.4. *([10]).Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$-metric space. If the sequence $\{\alpha_\theta\}$ in $\mathscr{M}$ converges to a point α, then the limit α is unique.*

Definition 2.8. . Let $(\mathscr{M}, \mathscr{A})$ be $\mathscr{A}_b$-metric space. A sequence $\{\alpha_\theta\}$ in $\mathscr{M}$ is called a Cauchy sequence, if $\mathscr{A}(\underbrace{\alpha_\theta, \alpha_\theta, \ldots, \alpha_\sigma}_{(\gamma-1)times}, \alpha_m) \to 0$ as $\sigma, m \to \infty$. That is, to each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $\sigma, m \geq N$, we have $A(\underbrace{\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma}_{(\gamma-1)times}, \alpha_m) \leq \varepsilon$.

Lemma 2.5. *([10]). Every convergent sequence in a $\mathscr{A}_b$-metric space is a Cauchy sequence.*

Definition 2.9. A $\mathscr{A}_b$-metric space $(\mathscr{M}, \mathscr{A})$ is said to be complete, if every Cauchy sequence in $\mathscr{M}$ is convergent.

Definition 2.10. ([7]). Let $(\mathscr{M}, \leq)$ be a partially ordered set and $\mathscr{O}, \mathscr{H} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be mappings. We say that $(\mathscr{O}, \mathscr{H})$ has the mixed weakly monotone property on $\mathscr{M}$, if for any $\alpha, \beta \in \mathscr{M}$, $\alpha \leq \mathscr{O}(\alpha, \beta)$, $\beta \geq \mathscr{O}(\beta, \alpha)$. Then

$$\mathscr{O}(\alpha, \beta) \leq \mathscr{H}((\mathscr{O}(\alpha, \beta), \mathscr{O}(\beta, \alpha)), \;\; \mathscr{O}(\beta, \alpha) \geq \mathscr{H}((\mathscr{O}(\beta, \alpha), \mathscr{O}(\alpha, \beta))$$

and $\alpha \leq \mathscr{H}(\alpha, \beta)$, $\beta \geq \mathscr{H}(\beta, \alpha)$ implies that

$$\mathscr{H}(\alpha, \beta) \leq \mathscr{O}((\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)), \;\; \mathscr{H}(\beta, \alpha) \geq \mathscr{O}((\mathscr{H}(\beta, \alpha), \mathscr{H}(\alpha, \beta)).$$

Definition 2.11. Let $\mathscr{M}$ be a non-empty set and $\mathscr{O}, \mathscr{H} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be maps on $\mathscr{M} \times \mathscr{M}$.
(i) A point $(\alpha, \beta) \in \mathscr{M} \times \mathscr{M}$ is called a coupled fixed pint of $\mathscr{O}$, if $\alpha = \mathscr{O}(\alpha, \beta)$ and $\beta = \mathscr{O}(\beta, \alpha)$.
(ii) A point $(\alpha, \beta) \in \mathscr{M} \times \mathscr{M}$ is said to be a common coupled fixed pint of $\mathscr{O}$ *and* $\mathscr{H}$, if $\alpha = \mathscr{O}(\alpha, \beta) = \mathscr{H}(\alpha, \beta)$ and $\beta = \mathscr{O}(\beta, \alpha) = \mathscr{H}(\beta, \alpha)$.

Note: We write D for D_A, when there is no confusion.

Note: (α, β) is said to be a Coupled coincidence point of $\mathscr{O}$ and $\mathscr{H}$, if $\mathscr{O}(\alpha, \beta) = \mathscr{H}(\alpha, \beta)$ and $\mathscr{O}(\beta, \alpha) = \mathscr{H}(\beta, \alpha)$.

We observe that a common coupled fixed pint of $\mathscr{O}$ and $\mathscr{H}$ is necessarily a Coupled coincidence point of $\mathscr{O}$ *and* $\mathscr{H}$.

3. Main Results

In this section, we generalize and extend the results of M. Abbas et al.[2] and N.Mlaiki et al.[10] .
Before going to the main results we introduce the notion of cross product $D = \mathscr{A}_b \times \mathscr{A}_b$ and study its properties.

Suppose $(\mathscr{M}, \leq, \mathscr{A})$ is a partially ordered complete $\mathscr{A}_b$-metric space. Define the partial order $'' \preceq''$ on $\mathscr{M} \times \mathscr{M}$ as follows: $(\alpha, \beta) \preceq (u, v)$ if $\alpha \leq u$ and $\beta \geq v$ for $(\alpha, \beta), (u, v) \in \mathscr{M}$, then $'' \preceq''$ is partial order on $\mathscr{M} \times \mathscr{M}$. Let $n(\geq 2)$ be a positive integer. Define $D : (\mathscr{M} \times \mathscr{M})^n \to [0, \infty)$ as follows

$$D((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)) = \mathscr{A}(\alpha_1, \alpha_2, \ldots, \alpha_n) + \mathscr{A}(\beta_1, \beta_2, \ldots, \beta_n).$$

Then from lemma 2.11, $(\mathscr{M} \times \mathscr{M}, \preceq, D)$ is a $\mathscr{A}_b$-metric space. Now we observe the following:

(1) A sequence $\{(\alpha_\sigma, \beta_\sigma)\}$ in $\mathscr{M} \times \mathscr{M} \to (\alpha, \beta) \iff \{\alpha_\sigma\} \to \alpha$ and $\{\beta_\sigma\} \to \beta$. Suppose that $\{(\alpha_\sigma, \beta_\sigma)\}$ in $\mathscr{M} \times \mathscr{M} \to (\alpha, \beta)$. That is,

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha, \beta)) \to 0 \text{ as } \sigma \to \infty.$$

Then

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{ as } \sigma \to \infty.$$

So

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) \to 0 \text{ as } \sigma \to \infty \text{ and } \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{ as } \sigma \to \infty.$$

Therefore $\{\alpha_\sigma\} \to \alpha$ and $\{\beta_\sigma\} \to \beta$. Conversely suppose that $\{\alpha_\sigma\} \to \alpha$ and $\{\beta_\sigma\} \to \beta$. That is, $\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) \to 0$ as $\sigma \to \infty$ and $\mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0$ as $\sigma \to \infty$. Then

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{ as } \sigma \to \infty.$$

So,

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha, \beta)) \to 0\text{as } \sigma \to \infty.$$

Therefore $\{(\alpha_\sigma, \beta_\sigma)\}$ in $\mathscr{M} \times \mathscr{M} \to (\alpha, \beta)$.

(2) $\{(\alpha_\sigma, \beta_\sigma)\}$ is a Cauchy sequence in $\mathscr{M} \times \mathscr{M} \iff \{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ are Cauchy sequence in $\mathscr{M}$. Suppose that $\{(\alpha_\sigma, \beta_\sigma)\}$ is a Cauchy sequence in $\mathscr{M} \times \mathscr{M}$. That is,

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha_m, \beta_m)) \to 0\text{as } \sigma, m \to \infty.$$

Then

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha_m) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) \to 0\text{as } \sigma, m \to \infty.$$

Hence,

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha_m) \to 0\text{as } \sigma, m \to \infty$$

and

$$\mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) \to 0 \text{as } \sigma, m \to \infty.$$

Therefore $\{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ are Cauchy sequence in $\mathscr{M}$. Conversely suppose that $\{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ are Cauchy sequence in $\mathscr{M}$. That is, $\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha_m) \to 0$ as $\sigma, m \to \infty$ and $\mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) \to 0$ as $\sigma, m \to \infty$. This implies that

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha_m) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) \to 0 \text{as } \sigma, m \to \infty.$$

So,

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha_m, \beta_m)) \to 0 \text{as } \sigma, m \to \infty.$$

Therefore $\{(\alpha_\sigma, \beta_\sigma)\}$ is a Cauchy sequence in $\mathscr{M} \times \mathscr{M}$.

(3) $\mathscr{M}$ is complete if and only if $\mathscr{M} \times \mathscr{M}$ is complete. Suppose $\{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ are complete in $\mathscr{M}$. That is, $\{\alpha_\sigma\} \to \alpha$ and $\{\beta_\sigma\} \to \beta$. Thus,

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) \to 0 \text{as } \sigma \to \infty$$

and

$$\mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{as } \sigma \to \infty.$$

So,

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, ...\alpha_\sigma, \alpha) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{as } \sigma \to \infty.$$

Then

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha, \beta)) \to 0 \text{as } \sigma \to \infty.$$

Thus $\{(\alpha_\sigma, \beta_\sigma)\}$ is complete in $\mathscr{M} \times \mathscr{M}$. Conversely, suppose that $\{(\alpha_\sigma, \beta_\sigma)\}$ is complete in $\mathscr{M} \times \mathscr{M}$. That is, $\{\alpha_\sigma, \beta_\sigma\} \to (\alpha, \beta)$. This implies that

$$D((\alpha_\sigma, \beta_\sigma), (\alpha_\sigma, \beta_\sigma), \ldots, (\alpha_\sigma, \beta_\sigma), (\alpha, \beta)) \to 0 \text{as } \sigma \to \infty.$$

So,

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{as } \sigma \to \infty.$$

Then

$$\mathscr{A}(\alpha_\sigma, \alpha_\sigma, \ldots, \alpha_\sigma, \alpha) \to 0 \text{as } \sigma \to \infty$$

and

$$\mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta) \to 0 \text{as } \sigma \to \infty.$$

Therefore $\{\alpha_\sigma\} \to \alpha$ and $\{\beta_\sigma\} \to \beta$. Now we prove our first main result.

Theorem 3.1. *Let $(\mathscr{M}, \leq, \mathscr{A})$ be a partially ordered, complete $\mathscr{A}_b$-metric space and $\mathscr{O}, \mathscr{H} : X^2 \to \mathscr{M}$ be two maps such that*
(i) the pair $(\mathscr{O}, \mathscr{H})$ has mixed weakly monotone property on $\mathscr{M}$ and there exists $\alpha_0, \beta_0 \in \mathscr{M}$ such that $\alpha_0 \leq \mathscr{O}(\alpha_0, \beta_0), \mathscr{O}(\beta_0, \alpha_0) \leq \beta_0$ or
$\alpha_0 \leq \mathscr{H}(\alpha_0, \beta_0), \mathscr{H}(\beta_0, \alpha_0) \leq \beta_0$,
(ii) there is an α such that $\alpha b((n-1)+b) < 1$ and

$$\mathscr{A}(\mathscr{O}(\alpha, \beta), \mathscr{O}(\alpha, \beta), \ldots, \mathscr{O}(\alpha, \beta), \mathscr{H}(u, v)) + \mathscr{A}(\mathscr{O}(\beta, \alpha), \mathscr{O}(\beta, \alpha), \ldots, \mathscr{O}(\beta, \alpha), \mathscr{H}(v, u))$$

$$\leq \theta \max\{D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(u,v)),$$
$$D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{H}(u,v),\mathscr{H}(v,u))), \tag{1}$$
$$D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\mathscr{H}(u,v),\mathscr{H}(v,u))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\}$$

for all $\alpha,\beta,u,v \in \mathscr{M}$ with $\alpha \leq u$ and $\beta \geq v$,
(iii) either $\mathscr{O}$ or $\mathscr{H}$ is continuous.

Then $\mathscr{O}$ and $\mathscr{H}$ have a coupled common fixed point in $\mathscr{M}$.

PROOF. Let (α_0,β_0) be a given point in $\mathscr{M}\times\mathscr{M}$, satisfying (i). Write $\alpha_1 = \mathscr{O}(\alpha_0,\beta_0), \beta_1 = \mathscr{O}(\beta_0,\alpha_0), \alpha_2 = \mathscr{H}(\alpha_1,\beta_1), \beta_2 = \mathscr{H}(\beta_1,\alpha_1)$. Define the sequences $\{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ inductively

$$\begin{aligned} \alpha_{\sigma+1} = \mathscr{O}(\alpha_\sigma,\beta_\sigma),\ \beta_{\sigma+1} = \mathscr{O}(\beta_\sigma,\alpha_\sigma) \\ \alpha_{\sigma+2} = \mathscr{H}(\alpha_{\sigma+1},\beta_{\sigma+1}),\ \beta_{\sigma+2} = \mathscr{H}(\beta_{\sigma+1},\alpha_{\sigma+1}) \end{aligned} \tag{2}$$

for all $\sigma \in \mathbb{N}$. Since $\alpha_0 \leq \mathscr{O}(\alpha_0,\beta_0),\ \beta_0 \geq \mathscr{O}(\beta_0,\alpha_0)$ and since $(\mathscr{O},\mathscr{H})$ has the mixed weakly monotone property, we have

$$\alpha_1 = \mathscr{O}(\alpha_0,\beta_0) \leq \mathscr{H}(\mathscr{O}(\alpha_0,\beta_0),\mathscr{O}(\beta_0,\alpha_0)) = \mathscr{H}(\alpha_1,\beta_1) = \alpha_2 \implies \alpha_1 \leq \alpha_2$$

and

$$\alpha_2 = \mathscr{H}(\alpha_1,\beta_1) \leq \mathscr{O}(\mathscr{H}(\alpha_1,\beta_1),\mathscr{H}(\beta_1,\alpha_1)) = \mathscr{O}(\alpha_2,\beta_2) = \alpha_3 \implies \alpha_2 \leq \alpha_3$$

also

$$\beta_1 = \mathscr{O}(\beta_0,\alpha_0) \geq \mathscr{H}(\mathscr{O}(\beta_0,\alpha_0),\mathscr{O}(\alpha_0,\beta_0)) = \mathscr{H}(\beta_1,\alpha_1) = \beta_2 \implies \beta_1 \geq \beta_2$$

and

$$\beta_2 = \mathscr{O}(\beta_1,\alpha_1) \geq \mathscr{O}(\mathscr{H}(\beta_1,\alpha_1),\mathscr{H}(\alpha_1,\beta_1)) = \mathscr{O}(\beta_2,\alpha_2) = \beta_3 \implies \beta_2 \geq \beta_3.$$

By induction,

$$\begin{aligned} i.e,\ \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq ...... \leq \alpha_\sigma \leq \alpha_{\sigma+1} \leq .... \\ \beta_0 \geq \beta_1 \geq \beta_2 ...... \geq \beta_\sigma \geq \beta_{\sigma+1} \geq, \end{aligned} \tag{3}$$

for all $\sigma \in \mathbb{N}$. Now we show that these sequences are Cauchy. Define $D_\sigma : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ by

$$\begin{aligned} D_\sigma &= D((\alpha_\sigma,\beta_\sigma),(\alpha_\sigma,\beta_\sigma),\ldots,(\alpha_\sigma,\beta_\sigma),(\alpha_{\sigma+1},\beta_{\sigma+1})) \\ &= \mathscr{A}(\alpha_\sigma,\alpha_\sigma,\ldots,\alpha_\sigma,\alpha_{\sigma+1}) + \mathscr{A}(\beta_\sigma,\beta_\sigma,\ldots,\beta_\sigma,\beta_{\sigma+1}) \end{aligned}$$

for all $\alpha_i, \beta_i \in \mathscr{M}, i, j = 1, 2, ...n$. Now

$$\begin{aligned}
D_{2\sigma+1} &= \mathscr{A}(\alpha_{2\sigma+1}, \alpha_{2\sigma+1}, \ldots, \alpha_{2\sigma+1}, \alpha_{2\sigma+2}) + \mathscr{A}(\beta_{2\sigma+1}, \beta_{2\sigma+1}, \ldots, \beta_{2\sigma+1}, \beta_{2\sigma+2})\\
&= \mathscr{A}(\mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma}), \mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma})...\mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma}), \mathscr{H}(\alpha_{2\sigma+1}, \beta_{2\sigma+1}))\\
&\quad + \mathscr{A}(\mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma}), \mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma})...\mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma}), \mathscr{H}(\beta_{2\sigma+1}, \alpha_{2\sigma+1}))\\
&\leq \theta \max\{D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&\qquad D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma}), \mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma}))),\\
&\qquad D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}),\\
&\qquad\quad (\mathscr{H}(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), \mathscr{H}(\beta_{2\sigma+1}, \alpha_{2\sigma+1}))),\\
&\qquad D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}),\\
&\qquad\quad (\mathscr{H}(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), \mathscr{H}(\beta_{2\sigma+1}, \alpha_{2\sigma+1}))),\\
&\qquad D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}),\\
&\qquad\quad (\mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma}), \mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma})))\}
\end{aligned}$$

From 2,

$$\begin{aligned}
D_{2\sigma+1} \leq \theta\ max\{&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2})),\\
&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2})),\\
&D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}))\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
= \theta\ max\{&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2})),\\
&D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2}))\}
\end{aligned}$$

by lemma 2.2,

$$\begin{aligned}
&\leq \theta\ max\{D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&\qquad D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1})),\\
&\qquad D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2})),\\
&\qquad (n-1)bD((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}))\\
&\qquad + b^2D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2}))\}\\
&= \theta\{(n-1)b[D((\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma}, \beta_{2\sigma}), ...(\alpha_{2\sigma}, \beta_{2\sigma}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}))]\\
&\qquad + b^2[D((\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+1}, \beta_{2\sigma+1}), ...(\alpha_{2\sigma+1}, \beta_{2\sigma+1}), (\alpha_{2\sigma+2}, \beta_{2\sigma+2}))]\}
\end{aligned}$$

$$= \theta\{(n-1)b[\mathscr{A}(\alpha_{2\sigma},\alpha_{2\sigma},\ldots,\alpha_{2\sigma},\alpha_{2\sigma+1})+\mathscr{A}(\beta_{2\sigma},\beta_{2\sigma},\ldots,\beta_{2\sigma},\beta_{2\sigma+1})] + b^2[\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2})+\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})]\} \tag{5}$$

Similarly we get,

$$\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})+\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2}) \leq \theta\{(n-1)b[\mathscr{A}(\beta_{2\sigma},\beta_{2\sigma},\ldots,\beta_{2\sigma},\beta_{2\sigma+1})+\mathscr{A}(\alpha_{2\sigma},\alpha_{2\sigma},\ldots,\alpha_{2\sigma},\alpha_{2\sigma+1})] + b^2[\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})+\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2})]\} \tag{6}$$

From 5 and 6 we have,

$$2D_{2\sigma+1} = 2[\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2})+\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})] \leq 2\theta\{(n-1)b[\mathscr{A}(\alpha_{2\sigma},\alpha_{2\sigma},\ldots,\alpha_{2\sigma},\alpha_{2\sigma+1})+\mathscr{A}(\beta_{2\sigma},\beta_{2\sigma},\ldots,\beta_{2\sigma},\beta_{2\sigma+1})] + b^2[\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2})+\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})]\}$$

Therefore

$$D_{2\sigma+1} \leq \theta\{(n-1)b[\mathscr{A}(\alpha_{2\sigma},\alpha_{2\sigma},\ldots,\alpha_{2\sigma},\alpha_{2\sigma+1})+\mathscr{A}(\beta_{2\sigma},\beta_{2\sigma},\ldots,\beta_{2\sigma},\beta_{2\sigma+1})] + b^2[\mathscr{A}(\alpha_{2\sigma+1},\alpha_{2\sigma+1},\ldots,\alpha_{2\sigma+1},\alpha_{2\sigma+2})+\mathscr{A}(\beta_{2\sigma+1},\beta_{2\sigma+1},\ldots,\beta_{2\sigma+1},\beta_{2\sigma+2})]\} \tag{7}$$

This implies that

$$(1-\theta b^2)D_{2\sigma+1} \leq \theta(n-1)bD_{2\sigma}$$

and therefore

$$D_{2\sigma+1} \leq \frac{\theta(n-1)b}{1-\theta b^2}D_{2\sigma}. \tag{8}$$

Put $\zeta = \frac{\theta(n-1)b}{1-\theta b^2}$, then $0<\zeta<1$. From 8, we have

$$D_{2\sigma+1} \leq \zeta D_{2\sigma}.$$

Similarly, we can show that

$$D_{2\sigma+2} \leq \zeta D_{2\sigma+1},$$

for $\sigma = 0,1,2,\ldots.$ Hence

$$D_{\sigma+1} \leq \zeta D_{\sigma}.$$

Therefore

$$D_{\sigma+1} \leq \zeta^{\sigma+1}D_0. \tag{9}$$

Define

$$D_{\sigma,l} = D(\underbrace{(\alpha_\sigma,\beta_\sigma),(\alpha_\sigma,\beta_\sigma),\ldots,(\alpha_\sigma,\beta_\sigma)}_{(n-1)-times},(\alpha_l,\beta_l))$$
$$= \mathscr{A}(\underbrace{\alpha_\sigma,\alpha_\sigma,\ldots,\alpha_\sigma}_{(n-1)-times},\alpha_l)+\mathscr{A}(\underbrace{\beta_\sigma,\beta_\sigma,\ldots,\beta_\sigma}_{(n-1)-times},\beta_l)$$

Now we have to show that $D_{\sigma,l}$ is a Cauchy sequence. By Lemma 2.2, for all $\sigma, m \in \mathbb{N},\ \sigma \leq m$, we have

$$\begin{aligned} D_{\sigma+1,m+1} &= \mathscr{A}(\alpha_{\sigma+1}, \alpha_{\sigma+1}, \ldots, \alpha_{\sigma+1}, \alpha_{m+1}) + \mathscr{A}(\beta_{\sigma+1}, \beta_{\sigma+1}, \ldots, \beta_{\sigma+1}, \beta_{m+1}) \\ &\leq b(n-1)[\mathscr{A}(\alpha_{\sigma+1}, \alpha_{\sigma+1}, \ldots, \alpha_{\sigma+1}, \alpha_{\sigma+2}) + \mathscr{A}(\beta_{\sigma+1}, \beta_{\sigma+1}, \ldots, \beta_{\sigma+1}, \beta_{\sigma+2})] \\ &\quad + b^2[\mathscr{A}(\alpha_{\sigma+2}, \alpha_{\sigma+2}, \ldots, \alpha_{\sigma+2}, \alpha_{m+1}) + \mathscr{A}(\beta_{\sigma+2}, \beta_{\sigma+2}, \ldots, \beta_{\sigma+2}, \beta_{m+1})] \end{aligned}$$

$$\begin{aligned} &= b(n-1)D_{\sigma+1} + b^2 b(n-1)[\mathscr{A}(\alpha_{\sigma+2}, \alpha_{\sigma+2}, \ldots, \alpha_{\sigma+2}, \alpha_{\sigma+3}) \\ &\qquad + \mathscr{A}(\beta_{\sigma+2}, \beta_{\sigma+2}, \ldots, \beta_{\sigma+2}, \beta_{\sigma+3})] \\ &\quad + b^2 b^2[\mathscr{A}(\alpha_{\sigma+3}, \alpha_{\sigma+3}, \ldots, \alpha_{\sigma+3}, \alpha_{m+1}) + \mathscr{A}(\beta_{\sigma+3}, \beta_{\sigma+3}, \ldots, \beta_{\sigma+3}, \beta_{m+1})] \\ &= b(n-1)D_{\sigma+1} + b^3(n-1)D_{\sigma+2} + b^5(n-1)D_{\sigma+3} \\ &\qquad \vdots \\ &\quad + b^{2(m-k)-3}(n-1)[\mathscr{A}(\alpha_{m-1}, \alpha_{m-1}, \ldots, \alpha_{m-1}, \alpha_m) \\ &\qquad + \mathscr{A}(\beta_{m-1}, \beta_{m-1}, \ldots, \beta_{m-1}, \beta_m)] \\ &\quad + b^{2(m-k)-1}(n-1)[\mathscr{A}(\alpha_m, \alpha_m, \ldots, \alpha_m, \alpha_{m+1}) \\ &\qquad + \mathscr{A}(\beta_m, \beta_m, \ldots, \beta_m, \beta_{m+1})] \end{aligned}$$

From 9,

$$\begin{aligned} & D_{\sigma+1,m+1} \leq b(n-1)[\zeta^{\sigma+1} + b^2\zeta^{\sigma+2} + b^4\zeta^{\sigma+3} ... + b^{2(m-k)-2}\zeta^m]D_0 \\ \implies & D_{\sigma+1,m+1} \leq b(n-1)\zeta^{\sigma+1}[1 + b^2\zeta + (b^2\zeta)^2 + ... + (b^2\zeta)^{(m-k-1)}]D_0 \\ & \qquad = b(n-1)\zeta^{\sigma+1}[1 + \theta + \theta^2 + ... + \theta^{(m-k-1)}]D_0 \\ & \qquad \leq b(n-1)\zeta^{\sigma+1}(\frac{1}{1-\theta})D_0 \end{aligned}$$

Since $0 < \zeta < 1$, we have

$$\lim_{\sigma,m\to\infty} \mathscr{A}(\alpha_\sigma, \alpha_\sigma, ...\alpha_\sigma, \alpha_m) + \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) = 0.$$

That is

$$\lim_{\sigma,m\to\infty} \mathscr{A}(\alpha_\sigma, \alpha_\sigma, ...\alpha_\sigma, \alpha_m) = \lim_{\sigma,m\to\infty} \mathscr{A}(\beta_\sigma, \beta_\sigma, \ldots, \beta_\sigma, \beta_m) = 0.$$

Therefore $\{\alpha_\sigma\}$ and $\{\beta_\sigma\}$ are both Cauchy sequences in $\mathscr{M}$. By the completeness of X, there exists $\alpha, y \in \mathscr{M}$ such that $\alpha_\sigma \to \alpha$ and $\beta_\sigma \to \beta$ as $\sigma \to \infty$. Therefore $D_{\sigma,l}$ is a Cauchy sequence.

Now we show that (α, β) is a coupled fixed point of $\mathscr{O}$ and $\mathscr{H}$. Without loss of generality, we may suppose that $\mathscr{O}$ is continuous, we have

$$x = \lim_{\sigma\to\infty} \alpha_{2\sigma+1} = \lim_{\sigma\to\infty} \mathscr{O}(\alpha_{2\sigma}, \beta_{2\sigma}) = \mathscr{O}(\lim_{\sigma\to\infty} \alpha_{2\sigma}, \lim_{\sigma\to\infty} \beta_{2\sigma}) = \mathscr{O}(\alpha, \beta)$$

and

$$y = \lim_{\sigma\to\infty} \beta_{2\sigma+1} = \lim_{\sigma\to\infty} \mathscr{O}(\beta_{2\sigma}, \alpha_{2\sigma}) = \mathscr{O}(\lim_{\sigma\to\infty} \beta_{2\sigma}, \lim_{\sigma\to\infty} \alpha_{2\sigma}) = \mathscr{O}(\beta, \alpha)$$

Thus (α, β) is a coupled fixed point of $\mathscr{O}$. From 1, we have

$$\begin{aligned}
&\mathscr{A}(\alpha, \alpha, \dots, \alpha, \mathscr{H}(\alpha, \beta)) + \mathscr{A}(\beta, \beta, \dots, \beta, \mathscr{H}(\beta, \alpha))\\
&= \mathscr{A}(\mathscr{O}(\alpha, \beta), \mathscr{O}(\alpha, \beta), \dots, \mathscr{O}(\alpha, \beta), \mathscr{H}(\alpha, \beta))\\
&\ + \mathscr{A}(\mathscr{O}(\beta, \alpha), \mathscr{O}(\beta, \alpha), \dots, \mathscr{O}(\beta, \alpha), \mathscr{H}(\beta, \alpha))\\
&\leq \theta \max\{D((\alpha, \beta), (\alpha, \beta), ...(\alpha, \beta), (\alpha, \beta)),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{O}(\alpha, \beta), \mathscr{O}(\beta, \alpha))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{O}(\alpha, \beta), \mathscr{O}(\beta, \alpha)))\}\\
&= \theta \max\{0, D((\alpha, \beta), (\alpha, \beta), ...(\alpha, \beta), (\alpha, \beta)),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\alpha, \beta))\}\\
&= \theta\{D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)))\}\\
&\leq \theta b((\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)), (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)), \dots, (\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)), (\alpha, \beta)).
\end{aligned}$$

Since $\theta b < 1$, we have $(\mathscr{H}(\alpha, \beta), \mathscr{H}(\beta, \alpha)) = (\alpha, \beta)$. This implies that $\mathscr{H}(\alpha, \beta) = \alpha$ and $\mathscr{H}(\beta, \alpha) = \beta$. Therefore (α, β) is a coupled fixed point of $\mathscr{H}$. Thus (α, β) is a coupled common fixed point of $\mathscr{O}$ and $\mathscr{H}$. ☐

Theorem 3.2. *Let $(\mathscr{M}, \leq, \mathscr{M})$ be a partially ordered, complete $\mathscr{A}_b$-metric space and $f, g : X^2 \to \mathscr{M}$ be two maps such that*
(i) the pair $(\mathscr{O}, \mathscr{H})$ has mixed weakly monotone property on X and there exists $\alpha_0, \beta_0 \in \mathscr{M}$ such that $\alpha_0 \leq \mathscr{O}(\alpha_0, \beta_0), \mathscr{O}(\beta_0, \alpha_0) \leq \beta_0$ or
$\alpha_0 \leq \mathscr{H}(\alpha_0, \beta_0), \mathscr{H}(\beta_0, \alpha_0) \leq \beta_0$,
(ii) there is an θ such that $\theta b((n-1) + b) < 1$ and

$$\mathscr{A}(\mathscr{O}(\alpha, \beta), \mathscr{O}(\alpha, \beta), \dots, \mathscr{O}(\alpha, \beta), \mathscr{H}(u, v)) + \mathscr{A}(\mathscr{O}(\beta, \alpha), \mathscr{O}(\beta, \alpha), \dots, \mathscr{O}(\beta, \alpha), \mathscr{H}(v, u))$$

$$\begin{aligned}
&\leq \theta \max\{D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (u, v)),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{O}(\alpha, \beta), \mathscr{O}(\beta, \alpha))),\\
&\qquad D((u, v), (u, v), ..., (u, v), (\mathscr{H}(u, v), \mathscr{H}(v, u))),\\
&\qquad D((\alpha, \beta), (\alpha, \beta), ..., (\alpha, \beta), (\mathscr{H}(u, v), \mathscr{H}(v, u))),\\
&\qquad D((u, v), (u, v), ..., (u, v), (\mathscr{O}(\alpha, \beta), \mathscr{O}(\beta, \alpha)))\}
\end{aligned}$$

for all $\alpha, \beta, u, v \in \mathscr{M}$ with $\alpha \leq u$ and $\beta \geq v$,
(iii) X has the following properties
(a). if $\{\alpha_\sigma\}$ is an increasing sequence with $\alpha_\sigma \to \alpha$, then $\alpha_\sigma \leq \alpha$ for all $\sigma \in \mathbb{N}$,
(b). if $\{\beta_\sigma\}$ is a decreasing sequence with $\beta_\sigma \to \beta$, then $\beta \leq \beta_\sigma$ for all $\sigma \in \mathbb{N}$.
Then f and g have coupled common fixed points in $\mathscr{M}$.

PROOF. Suppose $\mathscr{M}$ satisfies (a) and (b), by 3 we get $\alpha_\sigma \leq \alpha$ and $\beta_\sigma \geq \beta$ for all $\sigma \in \mathbb{N}$. Applying lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\\
&\leq b(n-1)D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha_{2\sigma+2},\beta_{2\sigma+2})\\
&\quad + b^2 D((\alpha_{2\sigma+2},\beta_{2\sigma+2}),(\alpha_{2\sigma+2},\beta_{2\sigma+2}),...,(\alpha_{2\sigma+2},\beta_{2\sigma+2}),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\\
&= b(n-1)D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\alpha_{2\sigma+2},\beta_{2\sigma+2}))\\
&\quad + b^2 D((\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1})),(\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1})),\\
&\quad ...,(\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1})),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))).
\end{aligned}$$

Then

$$\begin{aligned}
&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\\
&\leq b(n-1)[\mathscr{A}(\alpha,\alpha,...,\alpha,\alpha_{2\sigma+2})+\mathscr{A}(\beta,\beta,...,\beta,\beta_{2\sigma+2})]\\
&\quad + b^2 A[\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),...,\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{O}(\alpha,\beta)]\\
&\quad + b^2 A[\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1}),\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1}),...,\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1}),\mathscr{O}(\beta,\alpha)].
\end{aligned} \tag{10}$$

By 1, we get

$$\begin{aligned}
&\mathscr{A}((\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1})),(\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1})),...,(\mathscr{H}(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\mathscr{O}(\alpha,\beta))\\
&\quad + \mathscr{A}((\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1})),(\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1})),...(\mathscr{H}(\beta_{2\sigma+1},\alpha_{2\sigma+1}),(\mathscr{O}(\beta,\alpha))\\
&\leq \theta\max\{D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),...,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha,\beta)),\\
&\qquad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\ldots,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\\
&\qquad\qquad \mathscr{H}((\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}((\beta_{2\sigma+1},\alpha_{2\sigma+1})))),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\
&\qquad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\ldots,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),\mathscr{H}((\alpha_{2\sigma+1},\beta_{2\sigma+1}),\mathscr{H}((\beta_{2\sigma+1},\alpha_{2\sigma+1}))))\}\\
&= \theta\max\{D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\ldots,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha,\beta)),\\
&\qquad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\ldots,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+2},\beta_{2\sigma+2})),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\
&\qquad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),\ldots,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(\alpha_{2\sigma+2},\beta_{2\sigma+2}))\}.
\end{aligned}$$

From 8 and 9

$$\begin{aligned}&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\\&\leq b(n-1)[(\mathscr{A}(\alpha,\alpha,...,\alpha,\alpha_{2\sigma+2})+\mathscr{A}(\beta,\beta,\ldots,\beta,\beta_{2\sigma+2}))]\\&\quad+b^2\theta\max\{D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),...,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha,\beta)),\\&\quad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),...,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+2},\beta_{2\sigma+2})),\\&\quad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\&\quad D((\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\alpha_{2\sigma+1},\beta_{2\sigma+1}),...,(\alpha_{2\sigma+1},\beta_{2\sigma+1}),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\&\quad D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(\alpha_{2\sigma+2},\beta_{2\sigma+2}))\}\end{aligned}$$

Taking the limit as $\sigma\to\infty$ in 10, we obtain

$$\begin{aligned}&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\\&\leq b(n-1)[\mathscr{A}(\alpha,\alpha,...,\alpha,\alpha)+\mathscr{A}(\beta,\beta,...,\beta,\beta)]\\&\quad+b^2\theta\max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta)),D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta)),\\&\qquad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\&\qquad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\&\qquad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta))\}\\&=b^2\theta\ D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))).\end{aligned}$$

Since $b^2\theta<1$, we have

$$D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))=0.$$

So, $(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))=(\alpha,\beta)$. That is, $\mathscr{O}(\alpha,\beta)=x$ and $\mathscr{O}(\beta,\alpha)=y$. Therefore (α,β) is a coupled fixed point of $\mathscr{O}$. Similarly, we can show that $\mathscr{H}(\alpha,\beta)=x$ and $\mathscr{H}(\beta,\alpha)=y$. Hence $\mathscr{O}(\alpha,\beta)=x=\mathscr{H}(\alpha,\beta)$ and $\mathscr{O}(\beta,\alpha)=y=\mathscr{H}(\beta,\alpha)$. Thus (α,β) is a coupled common fixed point of $\mathscr{O}$ and $\mathscr{H}$. □

Theorem 3.3. *Suppose that Theorem 3.1 satisfied, if further $\{\alpha_n\}$ is an increasing sequence with $\alpha_n\to\alpha$ and $\alpha_n\leq u$ for each n, then $\alpha\leq u$. Then f and g have a unique coupled common fixed points. Further more, any fixed point of $\mathscr{O}$ is a fixed point of $\mathscr{H}$, and conversely.*

Proof. Suppose the given condition holds. Let (α,β) and $(u,v)\in\mathscr{M}\times\mathscr{M}$, there exist $(\alpha^*,\beta^*)\in\mathscr{M}\times\mathscr{M}$, that is, comparable to (α,β) and (u,v). Then

$$\begin{aligned}&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v))=\mathscr{A}(\alpha,\alpha,...,\alpha,u)+\mathscr{A}(\beta,\beta,...\beta,u)\\&=\mathscr{A}(\mathscr{O}(\alpha,\beta),\mathscr{O}(\alpha,\beta),...,\mathscr{O}(\alpha,\beta),\mathscr{H}(u,v))+\mathscr{A}(\mathscr{O}(\beta,\alpha),\mathscr{O}(\beta,\alpha),...,\mathscr{O}(\beta,\alpha),\mathscr{H}(v,u))\end{aligned}$$

$$\leq \theta \max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{H}(u,v),\mathscr{H}(v,u))),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(u,v),\mathscr{H}(v,u))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\}$$
$$= \theta \max\{D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(u,v)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta)),$$
$$D((u,v),(u,v),...,(u,v),(u,v)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$D((u,v),(u,v),...,(u,v),(\alpha,\beta))\}$$
$$= \theta \max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$D((u,v),(u,v),...(u,v),(\alpha,\beta))\}$$
$$\leq \theta \max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$bD((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v))\}$$
$$= \theta b D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v))$$

Since $\alpha b < 1$, we have

$$D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(u,v)) = 0.$$

So, $(\alpha,\beta) = (u,v)$. Hence, $x = u$ and $y = v$. Suppose that (α,β) and (α^*,β^*) are the coupled common fixed points such that $\alpha \leq \alpha^*$ and $\beta \geq \beta^*$, then $\alpha = \alpha^*$ and $\beta = \beta^*$. Now, we have

$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha^*,\beta^*)) = \mathscr{A}(\alpha,\alpha,...,\alpha,x^*) + \mathscr{A}(\beta,\beta,...,\beta,\beta^*)$$
$$= \mathscr{A}(\mathscr{O}(\alpha,\beta),\mathscr{O}(\alpha,\beta),...,\mathscr{O}(\alpha,\beta),\mathscr{H}(\alpha^*,\beta^*))$$
$$+ \mathscr{A}(\mathscr{O}(\beta,\alpha),\mathscr{O}(\beta,\alpha),...,\mathscr{O}(\beta,\alpha),\mathscr{H}(\beta^*,x^*))$$
$$\leq \theta \max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha^*,\beta^*)),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),$$
$$D((\alpha^*,\beta^*),(\alpha^*,\beta^*),...,(\alpha^*,\beta^*),(\mathscr{H}(\alpha^*,\beta^*),\mathscr{H}(\beta^*,x^*))),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(\alpha^*,\beta^*),\mathscr{H}(\beta^*,x^*))),$$
$$D((\alpha^*,\beta^*),(\alpha^*,\beta^*),...,(\alpha^*,\beta^*),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\}$$
$$= \theta \max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta)),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha^*,\beta^*)),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha^*,\beta^*)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\alpha,\beta))\}$$
$$\leq \theta b\ D((\alpha^*,\beta^*),(\alpha^*,\beta^*),...,(\alpha^*,\beta^*),(\alpha,\beta)).$$

Since $\theta b < 1$,

$$D((\alpha^*,\beta^*),(\alpha^*,\beta^*),...,(\alpha^*,\beta^*),(\alpha,\beta)) = 0.$$

Thus, $(\alpha^*, \beta^*) = (\alpha, \beta)$. Hence, $\alpha = \alpha^*$ and $\beta = \beta^*$.

We show that any fixed point of $\mathscr{O}$ is a fixed point of $\mathscr{H}$, and conversely. That is, to show that (α, β) is a fixed point of f if and only if (α, β) is a fixed point of $\mathscr{H}$. Suppose that (α, β) is a coupled fixed point of $\mathscr{O}$.

$$
\begin{aligned}
&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)))\\
&= \mathscr{A}(\mathscr{O}(\alpha,\beta),\mathscr{O}(\alpha,\beta),...,\mathscr{O}(\alpha,\beta),\mathscr{H}(\alpha,\beta)) + \mathscr{A}(\mathscr{O}(\beta,\alpha),\mathscr{O}(\beta,\alpha),...\mathscr{O}(\beta,\alpha),\mathscr{H}(\beta,\alpha))\\
&\le \theta \max\{D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\alpha,\beta)), D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha))),\\
&D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha))),\\
&\qquad D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\}\\
&= \theta D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)))\\
&\le \theta b\ D((\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),\ldots,(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),(\alpha,\beta))
\end{aligned}
$$

Since $\theta b < 1$, we have
$D((\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),\ldots,(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)),(\alpha,\beta)) = 0.$

So,

$$(\mathscr{H}(\alpha,\beta),\mathscr{H}(\beta,\alpha)) = (\alpha,\beta).$$

Then

$$x = \mathscr{H}(\alpha,\beta) \ and \ y = \mathscr{H}(\beta,\alpha).$$

Therefore (α, β) is a coupled fixed point of $\mathscr{H}$, and conversely. □

Let $(\mathbb{R}, \le, \mathscr{M})$ be a partially ordered complete $\mathscr{A}_b$-metric space with $\mathscr{A}_b$-metric with index n, defined on $X = [0, +\infty)$ as $\mathscr{A}_b : X^n \to [0, +\infty)$ by

$$\mathscr{A}_b(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) = \sum_{i=1}^{n} \sum_{i<j} |\alpha_i - \alpha_j|^2$$

for all $\alpha_i \in \mathscr{M}$, $i = 1, 2, \ldots, n$. Then $(\mathscr{M}, \mathscr{A}_b)$ is an $\mathscr{A}_b$-metric space with $b = 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping defined by

$$\mathscr{O}(\alpha,\beta) = p\alpha + q\beta + n - 2, \qquad (p-1)^2 - q^2 = 0.$$

Let $\alpha_0 = 0$ and $\beta_0 = 0$. Thus $\alpha_0 = 0 \leq \mathscr{O}(0,0) = \mathscr{O}(\alpha_0, \beta_0)$, $\mathscr{O}(\beta_0, \alpha_0) = \mathscr{O}(0,0) \leq 0 = \beta_0$. Then the pair (f, f) has mixed weakly monotone property on $\mathbb{R}$

$$
\begin{aligned}
&\mathscr{A}(\mathscr{O}(\alpha,\beta), \mathscr{O}(\alpha,\beta), ...\mathscr{O}(\alpha,\beta), \mathscr{O}(u,v)) + \mathscr{A}(\mathscr{O}(\beta,\alpha), \mathscr{O}(\beta,\alpha), ...\mathscr{O}(\beta,\alpha), \mathscr{O}(v,u)) \\
&= |\mathscr{O}(\alpha,\beta) - \mathscr{O}(u,v)|^2 + |\mathscr{O}(\beta,\alpha) - \mathscr{O}(v,u)|^2 \\
&= |p\alpha + q\beta + n - 2 - (pu + qv + n - 2)|^2 \\
&\quad + |P\beta + q\alpha + n - 2 - (pv + qu + n - 2)|^2 \\
&= |p(\alpha - u) + q(\beta - v)|^2 + |p(\beta - v) + q(\alpha - u)|^2 \\
&= |p|^2 |\alpha - u|^2 + |q|^2 |\beta - v|^2 + |p|^2 |\beta - v|^2 + |q|^2 |\alpha - u|^2 \\
&\leq (|p|^2 + |q|^2)(|\alpha - u|^2) + (|p|^2 + |q|^2)(|\beta - v|^2) \\
&= (|p|^2 + |q|^2)(|\alpha - u|^2 + |\beta - v|^2) \\
&= (|p|^2 + |q|^2)\ D((\alpha,\beta), (\alpha,\beta), \ldots, ., (\alpha,\beta), (u,v)) \\
&\leq (|p|^2 + |q|^2)\ \max\{D((\alpha,\beta), (\alpha,\beta), ...(\alpha,\beta), (u,v)), \\
&\qquad D((\alpha,\beta), (\alpha,\beta), ..., (\alpha,\beta), (\mathscr{O}(\alpha,\beta), \mathscr{O}(\beta,\alpha))), \\
&\qquad D((u,v), (u,v), ..., (u,v), (\mathscr{O}(u,v), \mathscr{O}(v,u))), \\
&\qquad D((\alpha,\beta), (\alpha,\beta), ..., (\alpha,\beta), (\mathscr{O}(u,v), \mathscr{O}(v,u))), \\
&\qquad D((u,v), (u,v), ..., (u,v), (\mathscr{O}(\alpha,\beta), \mathscr{O}(\beta,\alpha)))\}.
\end{aligned}
$$

Put $n = 2$ and $b = 2$. Since $\theta b((n-1) + b) < 1 \implies \theta < \frac{1}{3}$. Then the contractive condition 1 is satisfied with $\theta = |p|^2 + |q|^2 < \frac{1}{3}$. Choose a and b such that $a = \frac{1}{18}$, $b = \frac{1}{45}$. Then $\theta = |p|^2 + |q|^2 = \frac{1}{18}^2 + \frac{1}{45}^2 = 0.00358 < \frac{1}{3}$, and more over $(0,0)$ is the unique coupled fixed point of $\mathscr{O}$.

Example 3.1. Let $(\mathbb{R}, \leq, \mathscr{M})$ be a partially ordered complete $\mathscr{A}_b$-metric space with $\mathscr{A}_b$-metric with index n, defined on $X = [0,\infty)$ as $\mathscr{A}_b : X^n \to [0,\infty)$ by $\mathscr{A}_b(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) = \sum_{i=1}^{n} \sum_{i<j} |\alpha_i - \alpha_j|^2$, for all $\alpha_i \in \mathscr{M}$, $i = 1, 2, \ldots, n$. Then $(\mathscr{M}, \mathscr{A}_b)$ is an $\mathscr{A}_b$-metric space with b=2.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two maps defined by $\mathscr{O}(\alpha,\beta) = \frac{4\alpha - 2\beta + 48n - 2}{48n}$ and $\mathscr{H}(\alpha,\beta) = \frac{6\alpha - 3\beta + 72n - 3}{72n}$. Then the pair $(\mathscr{O}, \mathscr{H})$ has mixed weakly monotone property on $\mathbb{R}$

$$\mathscr{A}(\mathscr{O}(\alpha,\beta),\mathscr{O}(\alpha,\beta),...\mathscr{O}(\alpha,\beta),\mathscr{H}(u,v))+\mathscr{A}(\mathscr{O}(\beta,\alpha),\mathscr{O}(\beta,\alpha),...\mathscr{O}(\beta,\alpha),\mathscr{H}(v,u))$$
$$=[\mathscr{O}(\alpha,\beta)-\mathscr{H}(u,v)]^2+[\mathscr{O}(\beta,\alpha)-\mathscr{H}(v,u)]^2$$
$$=\left[\left|\frac{4\alpha-2\beta+48n-2}{48n}-\frac{6u-3v+72n-3}{72n}\right|^2\right]$$
$$+\left[\left|\frac{4\beta-2\alpha+48n-2}{48n}-\frac{6v-3u+72n-3}{72n}\right|^2\right]$$
$$=\frac{1}{(24n)^2}[|2(\alpha-u)-(\beta-v)|^2+|2(\beta-v)-(\alpha-u)|^2]$$
$$\leq\frac{1}{(24n)^2}[9\,|\alpha-u|^2+9\,|\beta-v|^2]$$
$$\leq\frac{3}{24n}[|\alpha-u|^2+|\beta-v|^2]$$
$$=\frac{3}{24n}\ D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(u,v))$$
$$\leq\frac{(3)}{24n}\max\{D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(u,v)),$$
$$D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{H}(u,v),\mathscr{H}(v,u))),$$
$$D((\alpha,\beta),(\alpha,\beta),...,(\alpha,\beta),(\mathscr{H}(u,v),\mathscr{H}(v,u))),$$
$$D((u,v),(u,v),...,(u,v),(\mathscr{O}(\alpha,\beta),\mathscr{O}(\beta,\alpha)))\}$$

For $n=2$ and b=2, since $\alpha b((n-1)+b)<1 \implies \alpha<\frac{1}{3}$. Then the contractive condition 1 is satisfied with $\theta=\frac{1}{16}<\frac{1}{3}$, and also (1,1) is the unique coupled common fixed point of $\mathscr{O}$ and $\mathscr{H}$.

Corollary 3.4. *Let $(\mathscr{M},\leq,\mathscr{M})$ be a partially ordered complete $\mathscr{A}_b$-metric space and $f: X^2\to\mathscr{M}$ such that*
(i) f has mixed weakly monotone property on $\mathscr{M}$ and there exist $\alpha_0,\beta_0\in\mathscr{M}$ such that $\alpha_0\leq\mathscr{O}(\alpha_0,\beta_0),\mathscr{O}(\beta_0,\alpha_0)\leq\beta_0$,
(ii) there is an θ, such that $\theta<1$ and

$$\mathscr{A}(\mathscr{O}(\alpha,\beta),\mathscr{O}(\alpha,\beta),\ldots,\mathscr{O}(\alpha,\beta),\mathscr{O}(u,v))+\mathscr{A}(\mathscr{O}(\beta,\alpha),\mathscr{O}(\beta,\alpha),\ldots,\mathscr{O}(\beta,\alpha),\mathscr{O}(v,u))$$
$$\leq\theta\{D((\alpha,\beta),(\alpha,\beta),...(\alpha,\beta),(u,v))\} \tag{11}$$

for all $\alpha,\beta,u,v\in\mathscr{M}$ with $\alpha\leq u$ and $\beta\geq v$.
(iii) f is continuous or X has the following properties.
(a) if $\{\alpha_\sigma\}$ is an increasing sequence with $\alpha_\sigma\to\alpha$, then $\alpha_\sigma\leq\alpha$ for all $\sigma\in\mathbb{N}$,
(b) if $\{\beta_\sigma\}$ is a decreasing sequence with $\beta_\sigma\to\beta$, then $\beta\leq\beta_\sigma$ for all $\sigma\in\mathbb{N}$.
Then f has a coupled fixed point in $\mathscr{M}$.

4. Application to Volterra type integral equations

Volterra integral equations are quite in numerous research disciplines, including migration patterns, infectious transmission, and non-linear propagations. Volterra continued to work on integral equations early 1884, but it wasn't until 1896 that he focussed on them seriously. Du Bois-Reymond came up with the term "integral equation" in 1888. Lalesco, on the other hand, proposed the phrase Volterra integral equation around 1908. The study and utilization of ordinary differential equations are familiar among most mathematicians, physicists, and several other researchers. The authors were able to apply their expertise of ordinary differential equations to the theoretical and practical of further common challenges by using Volterra integral and operational nonlinear systems in almost the same framework.
Consider the following system of integral equations:

$$\begin{aligned} u(\Lambda) &= q(\Lambda) + \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,u(\chi)) + \mathscr{O}_2(\chi,v(\chi)))d\chi \\ v(\Lambda) &= q(\Lambda) + \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,v(\chi)) + \mathscr{O}_2(\chi,u(\chi)))d\chi \end{aligned} \tag{12}$$

where the space $\mathscr{M} = C([a,b],\mathbb{R})$ of continuous functions defined in $[a,b]$. Obviously, the space with the metric is given by

$$\mathscr{A}(u,u,,\ldots,u,v) = |u(\Lambda) - v(\Lambda)|^2,\ u,v \in C([a,b],\mathbb{R})$$

is a complete metric space. Let $\mathscr{M} = C([a,b],\mathbb{R})$ the natural partial order relation, that is, $u,v \in C([a,b],\mathbb{R})$, $u \leq v$ if and only if $u(\Lambda) \leq v(\Lambda),\ t \in [a,b]$.

Theorem 4.1. *Consider the Corollary 3.4 and assume that the following conditions are hold:*
(i) $\mathscr{O}_1, \mathscr{O}_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*
(ii) $q : [a,b] \to \mathbb{R}$ *is continuous;*
(iii) $\lambda : [a,b] \times \mathbb{R} \to [0,\infty)$ *is continuous;*
(iv) there exist $c > 0$ *and* $0 \leq \theta < 1$*, such that for all* $u,v \in \mathbb{R},\ v \geq u$,
$0 \leq \mathscr{O}_1(\chi,v) - \mathscr{O}_1(\chi,u) \leq c\theta(v-u)$
$0 \leq \mathscr{O}_2(\chi,v) - \mathscr{O}_2(\chi,u) \leq c\theta(v-u)$*;*
(v) assume that $c\ \int_a^b \lambda(\Lambda,\chi)d\chi \leq 1$*;*

(vi) there exist $\alpha_0, \beta_0 \in \mathscr{M}$ *such that*

$$\begin{aligned} \alpha_0(\Lambda) &\geq q(\Lambda) + \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}(\chi,\alpha_0(\chi)) + \mathscr{H}(\chi,\beta_0(\chi)))d\chi \\ \beta_0(\Lambda) &\leq q(\Lambda) + \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}(\chi,\beta_0(\chi)) + \mathscr{H}(\chi,\alpha_0(\chi)))d\chi. \end{aligned}$$

Then the system of integral equation 12 has a unique solution in $\mathscr{M} \times \mathscr{M}$ with $\mathscr{M} = C([a,b],\mathbb{R})$.

PROOF. Define the mapping $\mathscr{F} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ by

$$\mathscr{F}(u,v)(\Lambda) = q(\Lambda) + \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,u(\chi)) + \mathscr{O}_2(\chi,v(\chi)))d\chi \tag{13}$$

for all $u,v \in \mathscr{M}$ *and* $t \in [a,b]$. Now we have to show that all the conditions of Corollary 3.4 are satisfied. From (iv) of the Theorem 4.1, clearly $\mathscr{F}$ has mixed monotone property. For $\alpha, \beta, u, v \in \mathscr{M}$ with $\alpha \geq u$ and $\beta \leq v$, we have

$$\begin{aligned}
&\mathscr{A}(\mathscr{F}(\alpha,\beta),\mathscr{F}(\alpha,\beta),\ldots,\mathscr{F}(\alpha,\beta),\mathscr{F}(u,v)) + \mathscr{A}(\mathscr{F}(\beta,\alpha),\mathscr{F}(\beta,\alpha),\ldots,.,\mathscr{F}(\beta,\alpha),\mathscr{F}(v,u))\\
&= |\mathscr{F}(\alpha,\beta)(\Lambda) - \mathscr{F}(u,v)(\Lambda)|^2 + |\mathscr{F}(\beta,\alpha)(\Lambda) - \mathscr{F}(v,u)(\Lambda)|^2\\
&= \left|\int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,\alpha(\chi)) + \mathscr{O}_2(\chi,y(\chi)))d\chi - \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,u(\chi)) + \mathscr{O}_2(\chi,v(\chi)))^2 d\chi\right|^2\\
&\quad + \left|\int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,y(\chi)) + \mathscr{O}_2(\chi,\alpha(\chi)))d\chi - \int_a^b \lambda(\Lambda,\chi)(\mathscr{O}_1(\chi,v(\chi)) + \mathscr{O}_2(\chi,u(\chi)))d\chi\right|^2\\
&\leq \Bigg(\int_a^b |\mathscr{O}_1(\chi,\alpha(\chi)) - \mathscr{O}_1(\chi,u(\chi))|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi\\
&\quad + \int_a^b |\mathscr{O}_2(\chi,y(\chi)) - \mathscr{O}_2(\chi,v(\chi))|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi\\
&\quad + \int_a^b |\mathscr{O}_1(\chi,y(\chi)) - \mathscr{O}_1(\chi,v(\chi))|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi +\\
&\quad \int_a^b |\mathscr{O}_2(\chi,\alpha(\chi)) - \mathscr{O}_2(\chi,u(\chi))|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi\Bigg)\\
&\leq\ c^2\alpha^2\Bigg(\int_a^b |\alpha(\chi) - u(\chi)|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi + \int_a^b |y(\chi) - v(\chi)|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi\\
&\qquad + \int_a^b |y(\chi) - v(\chi)|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi + \int_a^b |\alpha(\chi) - v(\chi)|^2 \left|\lambda^2(\Lambda,\chi)\right| d\chi\Bigg)\\
&\qquad\qquad \leq \Bigg(|\alpha(\Lambda) - u(\Lambda)|^2 + |y(\Lambda) - v(\Lambda)|^2\\
&\qquad\qquad\qquad + |y(\Lambda) - v(\Lambda)|^2 + |\alpha(\Lambda) - u(\Lambda)|^2\Bigg)\ c^2\theta \int_a^b \left|\lambda^2(\Lambda,\chi)\right| d\chi\\
&\qquad\qquad \leq 2\Big(|\alpha(\Lambda) - u(\Lambda)|^2 + |y(\Lambda) - v(\Lambda)|^2\Big)\ c^2\theta \int_a^b \left|\lambda^2(\Lambda,\chi)\right| d\chi\\
&\qquad\qquad \leq 2\theta\ (\mathscr{A}(\alpha,x,...,\alpha,u) + \mathscr{A}(\beta,y,...,\beta,v))\\
&\qquad\qquad = 2\theta\ D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(u,v))
\end{aligned}$$

Therefore

$$\begin{aligned}&\mathscr{A}(\mathscr{F}(\alpha,\beta),\mathscr{F}(\alpha,\beta),\ldots,\mathscr{F}(\alpha,\beta),\mathscr{F}(u,v))\\&+\mathscr{A}(\mathscr{F}(\beta,\alpha),\mathscr{F}(\beta,\alpha),\ldots,\mathscr{F}(\beta,\alpha),\mathscr{F}(v,u))\\&\leq 2\theta D((\alpha,\beta),(\alpha,\beta),\ldots,(\alpha,\beta),(u,v))\end{aligned}$$

hence $\theta < \frac{1}{2} < 1$, which is the contractive condition in Corollary 3.4. Thus, F has a coupled fixed point in $\mathscr{M}$. That is, the system of integral equations has a solution. □

Acknowledgment

The first author (K. Ravibabu) is grateful to
(i) The editor and referees for their valuable suggestions and critical remarks for improving the presentation of this paper.
(ii) The authorities of G.M.R Institute of Technology, Rajam for providing necessary facilities to carry on this research and
(iii) JNT University, Kakinada for granting the necessary permissions to carry on this research.

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Received : *January* 2022
Accepted : *March* 2022