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Bicomplex valued bipolar metric spaces and fixed point theorems

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Abstract. The concept of bicomplex valued bipolar metric space is introduced in this article, and some properties are derived. Also, some fixed point results of contravariant maps satisfying rational inequalities are proved for bicomplex valued bipolar metric spaces.

1. Introduction

Let $\mathbb{C}_1$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}_1$. Define a partial order $\precsim$ on $\mathbb{C}_1$ as follows. $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(I) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(II) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(III) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(IV) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (I),(II) and (III) is satisfied, and we will write $z_1 \prec z_2$ if only (III) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$$
$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Let $\mathbb{C}_0$ and $\mathbb{C}_2$ be the set of all real and bicomplex numbers respectively. Bicomplex numbers are defined by C. Segre [14] as: $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$, where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that ${i_i}^2 = {i_2}^2 = -1$, and

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$i_1 i_2 = i_2 i_1$. We denote the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\tau : \tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

i.e., $\mathbb{C}_2 = \{\tau : \tau = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\}$, where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\tau = z_1 + i_2 z_2$ and $\nu = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\tau \pm \nu = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2(z_2 \pm w_2)$ and the product is $\tau.\nu = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + z_2 w_1)$.

An element $\nu = w_1 + i_2 w_2 \in \mathbb{C}_2$ is nonsingular if and only if $|{w_1}^2 + {w_2}^2| \neq 0$ and singular if and only if $|{w_1}^2 + {w_2}^2| = 0$. The inverse of ν is defined as $\nu^{-1} = \frac{w_1 - i_2 w_2}{{w_1}^2 + {w_2}^2}$.

A bicomplex number $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

is degenerated. In that case τ^{-1} exists and it is also degenerated.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to {\mathbb{C}_0}^+$ by $\|\tau\| = \|z_1 + i_2 z_2\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = ({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)^{\frac{1}{2}}$, where $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

Define a partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows. For $\tau = z_1 + i_2 z_2$ and $\nu = w_1 + i_2 w_2$ be any two bicomplex numbers. $\tau \precsim_{i_2} \nu$ if and only if $z_1 \precsim w_1$, and $z_2 \precsim w_2$. It follows that $\tau \precsim_{i_2} \nu$ if one of the following conditions is satisfied:

(i) $z_1 = w_1$, $z_2 = w_2$,
(ii) $z_1 \prec w_1$, $z_2 = w_2$,
(iii) $z_1 = w_1$, $z_2 \prec w_2$,
(iv) $z_1 \prec w_1$, $z_2 \prec w_2$.

In particular we will write $\tau \precnsim_{i_2} \nu$ if $\tau \precsim_{i_2} \nu$ and $\tau \neq \nu$ and one of (ii),(iii), and (iv) is satisfied, and we will write $\tau \prec \nu$ if only (iv) is satisfied. Note that

(I) $\tau \precsim_{i_2} \nu \Rightarrow \|\tau\| \leq \|\nu\|$,
(II) $\|\tau + \nu\| \leq \|\tau\| + \|\nu\|$,
(III) $\|a\tau\| = a\|\tau\|$, where a is a non negative real number,
(IV) $\|\tau\nu\| \leq \sqrt{2}\|\tau\|\|\nu\|$, and the equality holds only when atleast one of τ and ν is degenerated,
(V) $\|\tau^{-1}\| = \|\tau\|^{-1}$ if τ is a degenerated bicomplex number with $0 \prec \tau$,
(VI) $\|\frac{\tau}{\nu}\| = \frac{\|\tau\|}{\|\nu\|}$, if ν is a degenerated bicomplex number.

A. Azam et al introduced the concept of complex valued metric spaces in [1]. The notion of bicomplex valued metric spaces was introduced by J. Choi et al in [3], some properties were derived and common fixed point results for mappings satisfying a rational inequality were proved. There are many articles appeared for fixed point theory in bicomplex valued metric spaces, see [2, 4, 5, 6, 7, 13].

Definition 1.1. [1] Let G be a non empty set. A bicomplex valued metric is a mapping $d : G \times G \to \mathbb{C}_2$ satisfying the following axioms:

(i) $0 \precsim_{i_2} d(\vartheta, \varpi)$, $\forall\ \vartheta, \varpi \in G$,
(ii) $d(\vartheta, \varpi) = 0$ if and only if $\vartheta = \varpi$ in G,
(iii) $d(\vartheta, \varpi) = d(\varpi, \vartheta)$, $\forall\ \vartheta, \varpi \in G$,
(iv) $d(\vartheta, \varpi) \precsim_{i_2} d(\vartheta, \kappa) + d(\kappa, \varpi)$, $\forall\ \vartheta, \kappa, \varpi \in G$.

The pair (G, d) is called a bicomplex valued metric space.

A. Mutlu et al [11] introduced the notion of bipolar metric space to giving a new definition of distance measurement between the members of two separate sets. Bipolar metric space is a metric space generalization. Many articles are appearing for fixed point theory in bipolar metric spaces, see for example [8, 9, 10, 12, 15] and the references therein.

Definition 1.2. [11] Let G and H be two non empty sets. A bipolar metric is a mapping $D : G \times H \to [0, \infty)$ satisfying the following axioms:

(I) $D(\vartheta, \varpi) = 0 \Rightarrow \vartheta = \varpi$, whenever $(\vartheta, \varpi) \in G \times H$,
(II) $\vartheta = \varpi \Rightarrow D(\vartheta, \varpi) = 0$, whenever $(\vartheta, \varpi) \in G \times H$,
(III) $D(\vartheta, \varpi) = D(\varpi, \vartheta)$, $\forall\ \vartheta, \varpi \in G \cap H$,
(IV) $D(\vartheta_1, \varpi_2) \leq D(\vartheta_1, \varpi_1) + D(\vartheta_2, \varpi_1) + D(\vartheta_2, \varpi_2)$, $\forall\ \vartheta_1, \vartheta_2 \in G$, and $\varpi_1, \varpi_2 \in H$.

The triple (G, H, D) is called a bipolar metric space.

In this paper, we extend the domain of bicomplex valued metric to Cartesian product of two non-empty sets, and we present a new definition of bicomplex valued bipolar metric space that generalizes the notion of bicomplex valued metric space. Also, we derive some properties of bicomplex valued bipolar metric spaces. Moreover, we prove some fixed point results for contravariant maps satisfying various types of rational inequalities in bicomplex valued bipolar metric space.

2. Bicomplex valued bipolar metric spaces

Definition 2.1. Let G and H be two non empty sets. A bicomplex valued bipolar metric is a mapping $d : G \times H \to \mathbb{C}_2$ satisfying the following conditions:

(i) $0 \precsim_{i_2} d(\vartheta, \varpi)$, whenever $(\vartheta, \varpi) \in G \times H$,
(ii) $d(\vartheta, \varpi) = 0 \Rightarrow \vartheta = \varpi$, whenever $(\vartheta, \varpi) \in G \times H$,
(iii) $\vartheta = \varpi \Rightarrow d(\vartheta, \varpi) = 0$, whenever $(\vartheta, \varpi) \in G \times H$,
(iv) $d(\vartheta, \varpi) = d(\varpi, \vartheta)$, $\forall\ \vartheta, \varpi \in G \cap H$,
(v) $d(\vartheta_1, \varpi_2) \precsim_{i_2} d(\vartheta_1, \varpi_1) + d(\vartheta_2, \varpi_1) + d(\vartheta_2, \varpi_2)$, $\forall\ \vartheta_1, \vartheta_2 \in G$, and $\varpi_1, \varpi_2 \in H$.

The triple (G, H, d) is called a bicomplex valued bipolar metric space(or, BVBMS).

Remark 2.2. Let (G, H, d) be a BVBMS. If $G \cap H = \emptyset$, then (G, H, d) is called disjoint. The space (G, H, d) is said to be a joint if $G \cap H \neq \emptyset$. The sets H and G are called right pole and left pole of (G, H, d), respectively.

Example 2.3. Let $G = (0, \infty)$ and $H = (-\infty, 0]$. Let $d(\vartheta, \varpi) = (1 + i_1 + i_2 + i_1 i_2)|\vartheta - \varpi|$, where $(\vartheta, \varpi) \in G \times H$. Then (G, H, d) is a disjoint BVBMS.

Remark 2.4. Let (G, d) be a bicomplex valued metric space, then (G, G, d) is a BVBMS. Conversely, if (G, H, d) is a BVBMS such that $G = H$, then (G, d) is a bicomplex valued metric space.

Definition 2.5. Let (G, H, d) be a BVBMS. Where points of the sets H, G, and $G \cap H$ are called right, left, and central points respectively. A sequence that contains only right(or left, or central) points is called a right (or left, or central) sequence in (G, H, d).

Definition 2.6. Let (G, H, d) be a BVBMS. A left sequence $(\vartheta_n)_{n=1}^{\infty}$ converges to a right point ϖ(or $(\vartheta_n)_{n=1}^{\infty} \to \varpi$) if and only if for every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$(Natural numbers) such that $d(\vartheta_n, \varpi) \prec_{i_2} c, \forall\, n \geq n_0$. Also a right sequence $(\varpi_n)_{n=1}^{\infty}$ converges to a left point ϑ (or $(\varpi_n)_{n=1}^{\infty} \to \vartheta$) if and only if for every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(\vartheta, \varpi_n) \prec_{i_2} c$, $\forall\, n \geq n_0$. When it is given $(\kappa_n)_{n=1}^{\infty} \to \rho$ for a BVBMS (G, H, d) without precise data about the sequence, this means that either $(\kappa_n)_{n=1}^{\infty}$ is a right sequence and ρ is a left point, or $(\kappa_n)_{n=1}^{\infty}$ is a left sequence and ρ is a right point.

Lemma 2.1. *Let (G, H, d) be a BVBMS. Then a left sequence $(\vartheta_n)_{n=1}^{\infty}$ converges to a right point ϖ if and only if $\|d(\vartheta_n, \varpi)\| \to 0$, and also a right sequence $(\varpi_n)_{n=1}^{\infty}$ converges to a left point ϑ if and only if $\|d(\vartheta, \varpi_n)\| \to 0$.*

Proof. Let $(\vartheta_n)_{n=1}^{\infty}$ be a left sequence, and $(\vartheta_n)_{n=1}^{\infty} \to \varpi \in H$. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1 \frac{\epsilon}{2} + i_2 \frac{\epsilon}{2} + i_1 i_2 \frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $d(\vartheta_n, \varpi) \prec_{i_2} c$.

$$\|d(\vartheta_n, \varpi)\| < \|c\| = \epsilon,\ \forall\, n \geq n_0.$$

It follows that $\|d(\vartheta_n, \varpi)\| \to 0$ as $n \to \infty$. Conversely, suppose that $\|d(\vartheta_n, \varpi)\| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}_2$

$$\|z\| < \delta \Rightarrow z \prec_{i_2} c$$

For this δ, there exists an integer $n_0 \in \mathbb{N}$ such that

$$\|d(\vartheta_n, \varpi)\| < \delta,\ \forall\, n \geq n_0.$$

This means that $d(\vartheta_n, \varpi) \prec_{i_2} c, \forall n \geq n_0$. Hence $\vartheta_n \to \varpi \in H$.
Obviously, a right sequence $(\varpi_n)_{n=1}^{\infty}$ converges to a left point ϑ if and only if $\|d(\vartheta, \varpi_n)\| \to 0$ and this complete the proof. □

Lemma 2.2. *Let (G, H, d) be a BVBMS. If a central point is a limit of a sequence, then it is the unique limit of the sequence.*

Proof. Let $(\vartheta_n)_{n=1}^{\infty}$ be a left sequence, $(\vartheta_n)_{n=1}^{\infty} \to \vartheta \in G \cap H$, and $(\vartheta_n)_{n=1}^{\infty} \to \varpi \in H$. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $d(\vartheta_n, \vartheta) \prec_{i_2} \frac{c}{2}$, and $d(\vartheta_n, \varpi) \prec_{i_2} \frac{c}{2}$, and then

$$d(\vartheta, \varpi) \precsim_{i_2} d(\vartheta, \vartheta) + d(\vartheta_n, \vartheta) + d(\vartheta_n, \varpi) \prec_{i_2} 0 + \frac{c}{2} + \frac{c}{2}.$$

$$\|d(\vartheta, \varpi)\| \leq \|d(\vartheta, \vartheta) + d(\vartheta_n, \vartheta) + d(\vartheta_n, \varpi)\| < \|0 + \frac{c}{2} + \frac{c}{2}\| = \|c\| = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $d(\vartheta, \varpi) = 0$ which implies $\vartheta = \varpi$. □

Lemma 2.3. *Let (G, H, d) be a BVBMS. If a left sequence $(\vartheta_n)_{n=1}^{\infty}$ converges to ϖ and a right sequence $(\varpi_n)_{n=1}^{\infty}$ converges to ϑ, then $d(\vartheta_n, \varpi_n) \to d(\vartheta, \varpi)$ as $n \to \infty$.*

Proof. Let $(\vartheta_n)_{n=1}^{\infty} \to \varpi \in H$, and $(\varpi_n)_{n=1}^{\infty} \to \vartheta \in G$. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $d(\vartheta_n, \varpi) \prec_{i_2} \frac{c}{2}$, and $d(\vartheta, \varpi_n) \prec_{i_2} \frac{c}{2}$, then

$$d(\vartheta, \varpi) \precsim_{i_2} d(\vartheta, \varpi_n) + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi)$$

implies

$$d(\vartheta, \varpi) - d(\vartheta_n, \varpi_n) \precsim_{i_2} d(\vartheta, \varpi_n) + d(\vartheta_n, \varpi) \prec \frac{c}{2} + \frac{c}{2},$$

$$\|d(\vartheta_n, \varpi_n) - d(\vartheta, \varpi)\| \leq \|d(\vartheta, \varpi_n) + d(\vartheta_n, \varpi)\| < \|c\| = \epsilon, \forall n \geq n_0,$$

and hence $d(\vartheta_n, \varpi_n) \to d(\vartheta, \varpi)$ as $n \to \infty$. □

Definition 2.7. Let (G_1, H_1) and (G_2, H_2) be two bicomplex valued bipolar metric spaces, and $f : G_1 \cup H_1 \to G_2 \cup H_2$.

(i) If $f(G_1) \subseteq G_2$ and $f(H_1) \subseteq H_2$, then f is called a covariant map from (G_1, H_1) to (G_2, H_2), and we write $f : (G_1, H_1) \rightrightarrows (G_2, H_2)$.

(ii) If $f(G_1) \subseteq H_2$ and $f(H_1) \subseteq G_2$, then f is called a contravariant map from (G_1, H_1) to (G_2, H_2), and we write $f : (G_1, H_1) \rightleftarrows (G_2, H_2)$.

Remark 2.8. Suppose d_1, and d_2 be two bicomplex valued bipolar metrics on (G_1, H_1) and (G_2, H_2) respectively. We can also use the symbols $f : (G_1, H_1, d_1) \rightrightarrows (G_2, H_2, d_2)$ and $f : (G_1, H_1, d_1) \rightleftarrows (G_2, H_2, d_2)$ in the place of $f : (G_1, H_1) \rightrightarrows (G_2, H_2)$ and $f : (G_1, H_1) \rightleftarrows (G_2, H_2)$.

Definition 2.9. Let (G, H, d) be a BVBMS.

(i) A sequence (ϑ_n, ϖ_n) on the set $G \times H$ is called a bisequence on (G, H, d).

(ii) If both $(\vartheta_n)_{n=1}^{\infty}$ and $(\varpi_n)_{n=1}^{\infty}$ converges, then the bisequence (ϑ_n, ϖ_n) is called convergent. If both $(\vartheta_n)_{n=1}^{\infty}$ and $(\varpi_n)_{n=1}^{\infty}$ converges to a same point $\vartheta \in G \cap H$, then the bisequence is called biconvergent.

(iii) A bisequence (ϑ_n, ϖ_n) on (G, H, d) is called a Cauchy bisequence, if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there is an $n_0 \in \mathbb{N}$ such that $d(\vartheta_n, \varpi_{n+m}) \prec_{i_2} c$, $\forall$ $n \geq n_0$.

Lemma 2.4. *Let (G, H, d) be a BVBMS. Then (ϑ_n, ϖ_n) is a Cauchy bisequence if and only if $\|d(\vartheta_n, \varpi_{n+m})\| \to 0$ as $n \to \infty$.*

Proof. Let (ϑ_n, ϖ_n) is a Cauchy bisequence. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $d(\vartheta_n, \varpi_{n+m}) \prec_{i_2} c$.

$$\|d(\vartheta_n, \varpi_{n+m})\| < \|c\| = \epsilon, \ \forall\, n \geq n_0.$$

It follows that $\|d(\vartheta_n, \varpi_{n+m})\| \to 0$ as $n \to \infty$. Conversely, suppose that $\|d(\vartheta_n, \varpi_{n+m})\| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}_2$

$$\|z\| < \delta \Rightarrow z \prec_{i_2} c$$

For this δ, there exists an integer $n_0 \in \mathbb{N}$ such that

$$\|d(\vartheta_n, \varpi_{n+m})\| < \delta, \ \forall\, n \geq n_0.$$

This means that $d(\vartheta_n, \varpi_{n+m}) \prec_{i_2} c, \forall n \geq n_0$. Hence (ϑ_n, ϖ_n) is a Cauchy bisequence. □

Proposition 2.5. *Let (G, H, d) be a BVBMS. Then every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let (ϑ_n, ϖ_n) be a bisequence, which is biconvergent to a point $\vartheta \in G \cap H$. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $d(\vartheta_n, \vartheta) \prec_{i_2} \frac{c}{2}$, and for every $n \geq n_0$, $d(\vartheta, \varpi_{n+m}) \prec_{i_2} \frac{c}{2}$. Then we have

$$d(\vartheta_n, \varpi_{n+m}) \precsim_{i_2} d(\vartheta_n, \vartheta) + d(\vartheta, \vartheta) + d(\vartheta, \varpi_{n+m}) \prec_{i_2} \frac{c}{2} + 0 + \frac{c}{2}, \forall n \geq n_0.$$

$$\|d(\vartheta_n, \varpi_{n+m})\| \leq \|d(\vartheta_n, \vartheta) + d(\vartheta, \vartheta) + d(\vartheta, \varpi_{n+m})\| < \|\frac{c}{2} + 0 + \frac{c}{2}\| = \|c\| = \epsilon, \quad \forall n \geq n_0.$$

So (ϑ_n, ϖ_n) is a Cauchy bisequence. □

Proposition 2.6. *Let (G, H, d) be a BVBMS. Then every convergent Cauchy bisequence is biconvergent.*

Proof. Let (ϑ_n, ϖ_n) be a Cauchy bisequence such that $(\vartheta_n)_{n=1}^{\infty}$ convergent to ϖ in H and $(\varpi_n)_{n=1}^{\infty}$ convergent to ϑ in G. For a given real number $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(\vartheta_n, \varpi) \prec_{i_2} \frac{c}{3}$, $d(\vartheta, \varpi_{n+m}) \prec_{i_2} \frac{c}{3}$, for all $n \geq n_0$, and $d(\vartheta_n, \varpi_{n+m}) \prec_{i_2} \frac{c}{3}$, for all $n \geq n_0$. Then

$$d(\vartheta, \varpi) \precsim_{i_2} d(\vartheta, \varpi_{n+m}) + d(\vartheta_n, \varpi_{n+m}) + d(\vartheta_n, \varpi) \prec_{i_2} \frac{c}{3} + \frac{c}{3} + \frac{c}{3}, \forall n \geq n_0.$$

$$\|d(\vartheta, \varpi)\| \leq \|d(\vartheta, \varpi_{n+m}) + d(\vartheta_n, \varpi_{n+m}) + d(\vartheta_n, \varpi)\| < \|\frac{c}{3} + \frac{c}{3} + \frac{c}{3}\| = \|c\| = \epsilon, \quad \forall n \geq n_0.$$

Therefore $d(\vartheta, \varpi) = 0$ and so that $\vartheta = \varpi$. Then (ϑ_n, ϖ_n) is biconvergent. □

Definition 2.10. A BVBMS (G, H, d) is called complete, if every Cauchy bisequence is convergent, or equivalently, biconvergent.

3. Main results

In this section we shall prove some fixed point theorems of different types of contravariant mappings on BVBMS.

Theorem 3.1. *Let (G, H, d) be a complete BVBMS with degenerated $1 + d(\vartheta, \varpi)$ and $\|1 + d(\vartheta, \varpi)\| \neq 0$, whenever $(\vartheta, \varpi) \in G \times H$. If a contravariant map $f : (G, H, d) \rightleftarrows (G, H, d)$ satisfies*

$$d(f(\varpi), f(\vartheta)) \precsim_{i_2} \lambda d(\vartheta, \varpi) + \frac{\mu d(\vartheta, f(\vartheta)) d(f(\varpi), \varpi)}{1 + d(\vartheta, \varpi)},$$

whenever $(\vartheta, \varpi) \in G \times H$, for some $\lambda, \mu \in (0, 1)$ with $\lambda + \sqrt{2}\mu < 1$. Then the function $f : G \cup H \to G \cup H$ has a UFP.

Proof. Let $\vartheta_0 \in G$, $\varpi_0 = f(\vartheta_0) \in H$, and $\vartheta_1 = f(\varpi_0)$. Suppose, $\varpi_n = f(\vartheta_n)$ and $\vartheta_{n+1} = f(\varpi_n)$, for all $n \in \mathbb{N}$. Then (ϑ_n, ϖ_n) is a bisequence on (G, H, d). For all $n \in \mathbb{N}$, from

$$\begin{aligned} d(\vartheta_n, \varpi_n) &= d(f(\varpi_{n-1}), f(\vartheta_n)) \\ &\precsim_{i_2} \lambda d(\vartheta_n, \varpi_{n-1}) + \frac{\mu d(\vartheta_n, f(\vartheta_n)) d(f(\varpi_{n-1}), \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \\ &= \lambda d(\vartheta_n, \varpi_{n-1}) + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \\ \|d(\vartheta_n, \varpi_n)\| &\leq \left\| \lambda d(\vartheta_n, \varpi_{n-1}) + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \right\| \\ &\leq \lambda \|d(\vartheta_n, \varpi_{n-1})\| + \sqrt{2}\mu \|d(\vartheta_n, \varpi_n)\| \end{aligned}$$

we conclude that

$$\|d(\vartheta_n, \varpi_n)\| \leq \frac{\lambda}{1-\sqrt{2}\mu}\|d(\vartheta_n, \varpi_{n-1})\|,$$

and

$$\begin{aligned} d(\vartheta_n, \varpi_{n-1}) &= d(f(\varpi_{n-1}), f(\vartheta_{n-1})) \\ &\precsim_{i_2} \lambda d(\vartheta_{n-1}, \varpi_{n-1}) + \frac{\mu d(\vartheta_{n-1}, f(\vartheta_{n-1}))d(f(\varpi_{n-1}), \varpi_{n-1})}{1+d(\vartheta_{n-1}, \varpi_{n-1})} \\ &= \lambda d(\vartheta_{n-1}, \varpi_{n-1}) + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1})d(\vartheta_n, \varpi_{n-1})}{1+d(\vartheta_{n-1}, \varpi_{n-1})} \\ \|d(\vartheta_n, \varpi_{n-1})\| &\leq \left\|\lambda d(\vartheta_{n-1}, \varpi_{n-1}) + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1})d(\vartheta_n, \varpi_{n-1})}{1+d(\vartheta_{n-1}, \varpi_{n-1})}\right\| \\ &\leq \lambda\|d(\vartheta_{n-1}, \varpi_{n-1})\| + \sqrt{2}\mu\|d(\vartheta_n, \varpi_{n-1})\|, \end{aligned}$$

so that

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \frac{\lambda}{1-\sqrt{2}\mu}\|d(\vartheta_{n-1}, \varpi_{n-1})\|,$$

Therefore, by putting $\alpha = \frac{\lambda}{1-\sqrt{2}\mu}$, we have

$$\|d(\vartheta_n, \varpi_n)\| \leq \alpha^{2n}\|d(\vartheta_0, \varpi_0)\|$$

and

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \alpha^{2n-1}\|d(\vartheta_0, \varpi_0)\|.$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\vartheta_n, \varpi_m) &\precsim_{i_2} d(\vartheta_n, \varpi_n) + d(\vartheta_{n+1}, \varpi_n) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1})d(\vartheta_0, \varpi_0) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} \ldots \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m-1})d(\vartheta_0, \varpi_0) + d(\vartheta_m, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m})d(\vartheta_0, \varpi_0), \text{ if } m > n, \end{aligned}$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m})\|d(\vartheta_0, \varpi_0)\|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\vartheta_n, \varpi_m) \precsim_{i_2} (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1})d(\vartheta_0, \varpi_0),$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1})\|d(\vartheta_0, \varpi_0)\|.$$

By $\alpha \in (0, 1)$, $\|d(\vartheta_n, \varpi_m)\| \to 0$, as $n, m \to \infty$, we conclude that (ϑ_n, ϖ_n) is a Cauchy bisequence. Since (G, H, d) is complete, (ϑ_n, ϖ_n) converges, and biconverges to a point $\kappa \in G \cap H$. Hence, $f(\vartheta_n) = \varpi_n \to \kappa \in G \cap H$ as $n \to \infty$ implies

$d(f(\kappa), f(\vartheta_n)) \to d(f(\kappa), \kappa)$ as $n \to \infty$, by using Lemma 2.3. Also by taking the limit from

$$d(f(\kappa), f(\vartheta_n)) \precsim_{i_2} \lambda d(\vartheta_n, \kappa) + \frac{\mu d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)}{1 + d(\vartheta_n, \kappa)}$$

we obtain

$$\|d(f(\kappa), f(\vartheta_n))\| \leq \lambda \|d(\vartheta_n, \kappa)\| + \frac{\mu \|d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)\|}{\|1 + d(\vartheta_n, \kappa)\|},$$

as $n \to \infty$, we get $d(f(\kappa), \kappa) = 0$. Hence $f(\kappa) = \kappa$. Therefore κ is a fixed point of f.

If ρ is another fixed point of f, then $f(\rho) = \rho$, $\rho \in G \cap H$, and hence,

$$d(\kappa, \rho) = d(f(\kappa), f(\rho)) \precsim_{i_2} \lambda d(\kappa, \rho) + \frac{\mu d(\kappa, f(\kappa)) d(f(\rho), \rho)}{1 + d(\kappa, \rho)} \precsim_{i_2} \lambda d(\kappa, \rho).$$

Therefore $\|d(\kappa, \rho)\| = 0$ so that $\kappa = \rho$. So f has a UFP. □

The above Theorem generalizes a Corollary 5 of [1] and Corollary 3.2 of [2].

Example 3.1. Let $G = \{0, \frac{1}{2}, 2\}$ and $H = \{0, \frac{1}{2}\}$. Let $d(\vartheta, \varpi) = (1 + i_2)|\vartheta - \varpi|$, where $(\vartheta, \varpi) \in G \times H$. Then (G, H, d) is a complete BVBMS. Define a contravariant map $f : (G, H, d) \rightleftarrows (G, H, d)$ by $f(0) = 0$, $f(\frac{1}{2}) = 0$, and $f(2) = \frac{1}{2}$. Then, f satisfies the inequality $d(f(\varpi), f(\vartheta)) \precsim_{i_2} \lambda d(\vartheta, \varpi) + \frac{\mu d(\vartheta, f(\vartheta)) d(f(\varpi), \varpi)}{1 + d(\vartheta, \varpi)}$ for $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{6}$. By Theorem 3.1, f has a UFP zero in $G \cap H$.

Theorem 3.2. *Let (G, H, d) be a complete BVBMS with degenerated $1 + d(\vartheta, \varpi)$ and $\|1 + d(\vartheta, \varpi)\| \neq 0$, whenever $(\vartheta, \varpi) \in G \times H$. If a contravariant map $f : (G, H, d) \rightleftarrows (G, H, d)$ satisfies*

$$d(f(\varpi), f(\vartheta)) \precsim_{i_2} \lambda [d(\vartheta, f(\vartheta)) + d(f(\varpi), \varpi)] + \frac{\mu d(\vartheta, f(\vartheta)) d(f(\varpi), \varpi)}{1 + d(\vartheta, \varpi)},$$

whenever $(\vartheta, \varpi) \in G \times H$, for some $\lambda, \mu \in (0, 1)$ with $2\lambda + \sqrt{2}\mu < 1$. Then the function $f : G \cup H \to G \cup H$ has a UFP.

Proof. Let $\vartheta_0 \in G$, $\varpi_0 = f(\vartheta_0) \in H$, and $\vartheta_1 = f(\varpi_0)$. Suppose, $\varpi_n = f(\vartheta_n)$ and $\vartheta_{n+1} = f(\varpi_n)$, for all $n \in \mathbb{N}$. Then (ϑ_n, ϖ_n) is a bisequence on (G, H, d). For all $n \in \mathbb{N}$, from

$$\begin{aligned} d(\vartheta_n, \varpi_n) &= d(f(\varpi_{n-1}), f(\vartheta_n)) \\ &\precsim_{i_2} \lambda[d(\vartheta_n, f(\vartheta_n)) + d(f(\varpi_{n-1}), \varpi_{n-1})] \\ &\quad + \frac{\mu d(\vartheta_n, f(\vartheta_n)) d(f(\varpi_{n-1}), \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \\ &= \lambda[d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})] + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \\ \|d(\vartheta_n, \varpi_n)\| &\leq \left\| \lambda[d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})] + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_{n-1})} \right\| \\ &\leq \lambda \|[d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})]\| + \sqrt{2}\mu \|d(\vartheta_n, \varpi_n)\|, \end{aligned}$$

we conclude that

$$\|d(\vartheta_n, \varpi_n)\| \leq \frac{\lambda}{1 - \lambda - \sqrt{2}\mu} \|d(\vartheta_n, \varpi_{n-1})\|,$$

and

$$\begin{aligned} d(\vartheta_n, \varpi_{n-1}) &= d(f(\varpi_{n-1}), f(\vartheta_{n-1})) \\ &\precsim_{i_2} \lambda[d(\vartheta_{n-1}, f(\vartheta_{n-1})) + d(f(\varpi_{n-1}), \varpi_{n-1})] \\ &\quad + \frac{\mu d(\vartheta_{n-1}, f(\vartheta_{n-1})) d(f(\varpi_{n-1}), \varpi_{n-1})}{1 + d(\vartheta_{n-1}, \varpi_{n-1})} \\ &= \lambda[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})] + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1}) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_{n-1}, \varpi_{n-1})} \\ \|d(\vartheta_n, \varpi_{n-1})\| &\leq \left\| \lambda[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})] + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1}) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_{n-1}, \varpi_{n-1})} \right\| \\ &\leq \lambda \|[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})]\| + \sqrt{2}\mu \|d(\vartheta_n, \varpi_{n-1})\| \end{aligned}$$

so that

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \frac{\lambda}{1 - \lambda - \sqrt{2}\mu} \|d(\vartheta_{n-1}, \varpi_{n-1})\|,$$

Therefore, by putting $\alpha = \frac{\lambda}{1-\lambda-\sqrt{2}\mu}$, we have

$$\|d(\vartheta_n, \varpi_n)\| \leq \alpha^{2n} \|d(\vartheta_0, \varpi_0)\|$$

and

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \alpha^{2n-1} \|d(\vartheta_0, \varpi_0)\|.$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\vartheta_n, \varpi_m) &\precsim_{i_2} d(\vartheta_n, \varpi_n) + d(\vartheta_{n+1}, \varpi_n) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1}) d(\vartheta_0, \varpi_0) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} \ldots \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m-1}) d(\vartheta_0, \varpi_0) + d(\vartheta_m, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m}) d(\vartheta_0, \varpi_0), \text{ if } m > n, \end{aligned}$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m}) \|d(\vartheta_0, \varpi_0)\|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\vartheta_n, \varpi_m) \precsim_{i_2} (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1}) d(\vartheta_0, \varpi_0),$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1}) \|d(\vartheta_0, \varpi_0)\|.$$

By $\alpha \in (0,1)$, $\|d(\vartheta_n, \varpi_m)\| \to 0$, as $n, m \to \infty$, we conclude that (ϑ_n, ϖ_n) is a Cauchy bisequence. Since (G, H, d) is complete, (ϑ_n, ϖ_n) converges, and biconverges to a point $\kappa \in G \cap H$. Hence, $f(\vartheta_n) = \varpi_n \to \kappa \in G \cap H$ as $n \to \infty$ implies $d(f(\kappa), f(\vartheta_n)) \to d(f(\kappa), \kappa)$ as $n \to \infty$, by using Lemma 2.3. Also by taking the limit from

$$d(f(\kappa), f(\vartheta_n)) \precsim_{i_2} \lambda[d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)] + \frac{\mu d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)}{1 + d(\vartheta_n, \kappa)}$$

we obtain

$$\|d(f(\kappa), f(\vartheta_n))\| \leq \lambda[\|d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)\|] + \frac{\mu \|d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)\|}{\|1 + d(\vartheta_n, \kappa)\|},$$

as $n \to \infty$, we get $d(f(\kappa), \kappa) = 0$. Hence $f(\kappa) = \kappa$. Therefore κ is a fixed point of f.
If ρ is another fixed point of f, then $f(\rho) = \rho$, $\rho \in G \cap H$, and hence,

$$d(\kappa, \rho) = d(f(\kappa), f(\rho)) \precsim_{i_2} \lambda[d(\kappa, f(\kappa)) + d(f(\rho), \rho)] + \frac{\mu d(\kappa, f(\kappa)) d(f(\rho), \rho)}{1 + d(\kappa, \rho)}.$$

Therefore $\|d(\kappa, \rho)\| = 0$ so that $\kappa = \rho$. So f has a UFP. □

Theorem 3.3. *Let (G, H, d) be a complete BVBMS with degenerated $1 + d(\vartheta, f(\vartheta)) + d(f(\varpi), \varpi)$ and $\|1 + d(\vartheta, f(\vartheta)) + d(f(\varpi), \varpi)\| \neq 0$, whenever $(\vartheta, \varpi) \in G \times H$. If a contravariant map $f : (G, H, d) \rightleftarrows (G, H, d)$ satisfies*

$$d(f(\varpi), f(\vartheta)) \precsim_{i_2} \lambda[d(\vartheta, \varpi) + d(\vartheta, f(\vartheta)) + d(f(\varpi), \varpi)] + \frac{\mu d(\vartheta, f(\vartheta)) d(f(\varpi), \varpi)}{1 + d(\vartheta, f(\vartheta)) + d(f(\varpi), \varpi)},$$

whenever $(\vartheta, \varpi) \in G \times H$, for some $\lambda, \mu \in (0,1)$ with $3\lambda + \sqrt{2}\mu < 1$. Then the function $f : G \cup H \to G \cup H$ has a UFP.

PROOF. Let $\vartheta_0 \in G$, $\varpi_0 = f(\vartheta_0) \in H$, and $\vartheta_1 = f(\varpi_0)$. Suppose, $\varpi_n = f(\vartheta_n)$ and $\vartheta_{n+1} = f(\varpi_n)$, for all $n \in \mathbb{N}$. Then (ϑ_n, ϖ_n) is a bisequence on (G, H, d). For all $n \in \mathbb{N}$, from

$$\begin{aligned}
d(\vartheta_n, \varpi_n) &= d(f(\varpi_{n-1}), f(\vartheta_n)) \\
&\precsim_{i_2} \lambda[d(\vartheta_n, \varpi_{n-1}) + d(\vartheta_n, f(\vartheta_n)) + d(f(\varpi_{n-1}), \varpi_{n-1})] \\
&\quad + \frac{\mu d(\vartheta_n, f(\vartheta_n)) d(f(\varpi_{n-1}), \varpi_{n-1})}{1 + d(\vartheta_n, f(\vartheta_n)) + d(f(\varpi_{n-1}), \varpi_{n-1})} \\
&= \lambda[d(\vartheta_n, \varpi_{n-1}) + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})] \\
&\quad + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})} \\
\|d(\vartheta_n, \varpi_n)\| &\leq \Big\| \lambda[d(\vartheta_n, \varpi_{n-1}) + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})] \\
&\quad + \frac{\mu d(\vartheta_n, \varpi_n) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})} \Big\| \\
&\leq \lambda \|[d(\vartheta_n, \varpi_{n-1}) + d(\vartheta_n, \varpi_n) + d(\vartheta_n, \varpi_{n-1})]\| + \sqrt{2}\mu \|d(\vartheta_n, \varpi_n)\|
\end{aligned}$$

we conclude that

$$\|d(\vartheta_n, \varpi_n)\| \leq \frac{2\lambda}{1 - \lambda - \sqrt{2}\mu} \|d(\vartheta_n, \varpi_{n-1})\|,$$

and

$$\begin{aligned}
d(\vartheta_n, \varpi_{n-1}) &= d(f(\varpi_{n-1}), f(\vartheta_{n-1})) \\
&\precsim_{i_2} \lambda[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_{n-1}, f(\vartheta_{n-1})) + d(f(\varpi_{n-1}), \varpi_{n-1})] \\
&\quad + \frac{\mu d(\vartheta_{n-1}, f(\vartheta_{n-1})) d(f(\varpi_{n-1}), \varpi_{n-1})}{1 + d(\vartheta_{n-1}, f(\vartheta_{n-1})) + d(f(\varpi_{n-1}), \varpi_{n-1})} \\
&= \lambda[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})] \\
&\quad + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1}) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_{n-1}, f(\vartheta_{n-1})) + d(f(\varpi_{n-1}), \varpi_{n-1})}
\end{aligned}$$

$$\begin{aligned}
&\|d(\vartheta_n, \varpi_{n-1})\| \leq \|\lambda[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})]\| \\
&\quad + \left\| + \frac{\mu d(\vartheta_{n-1}, \varpi_{n-1}) d(\vartheta_n, \varpi_{n-1})}{1 + d(\vartheta_{n-1}, f(\vartheta_{n-1})) + d(f(\varpi_{n-1}), \varpi_{n-1})} \right\| \\
&\leq \lambda \|[d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_{n-1}, \varpi_{n-1}) + d(\vartheta_n, \varpi_{n-1})]\| + \sqrt{2}\mu \|d(\vartheta_n, \varpi_{n-1})\|
\end{aligned}$$

so that

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \frac{2\lambda}{1 - \lambda - \sqrt{2}\mu} \|d(\vartheta_{n-1}, \varpi_{n-1})\|,$$

Therefore, by putting $\alpha = \frac{2\lambda}{1-\lambda-\sqrt{2}\mu}$, we have

$$\|d(\vartheta_n, \varpi_n)\| \leq \alpha^{2n} \|d(\vartheta_0, \varpi_0)\|$$

and

$$\|d(\vartheta_n, \varpi_{n-1})\| \leq \alpha^{2n-1}\|d(\vartheta_0, \varpi_0)\|.$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\vartheta_n, \varpi_m) &\precsim_{i_2} d(\vartheta_n, \varpi_n) + d(\vartheta_{n+1}, \varpi_n) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1}) d(\vartheta_0, \varpi_0) + d(\vartheta_{n+1}, \varpi_m) \\ &\precsim_{i_2} \ldots \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m-1}) d(\vartheta_0, \varpi_0) + d(\vartheta_m, \varpi_m) \\ &\precsim_{i_2} (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m}) d(\vartheta_0, \varpi_0), \text{ if } m > n, \end{aligned}$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m})\|d(\vartheta_0, \varpi_0)\|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\vartheta_n, \varpi_m) \precsim_{i_2} (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1}) d(\vartheta_0, \varpi_0),$$

$$\|d(\vartheta_n, \varpi_m)\| \leq (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1})\|d(\vartheta_0, \varpi_0)\|.$$

By $\alpha \in (0, 1)$, $\|d(\vartheta_n, \varpi_m)\| \to 0$, as $n, m \to \infty$, we conclude that (ϑ_n, ϖ_n) is a Cauchy bisequence. Since (G, H, d) is complete, (ϑ_n, ϖ_n) converges, and biconverges to a point $\kappa \in G \cap H$. Hence, $f(\vartheta_n) = \varpi_n \to \kappa \in G \cap H$ as $n \to \infty$ implies $d(f(\kappa), f(\vartheta_n)) \to d(f(\kappa), \kappa)$ as $n \to \infty$, by using Lemma 2.3. Also by taking the limit from

$$d(f(\kappa), f(\vartheta_n)) \precsim_{i_2} \lambda[d(\vartheta_n, \kappa) + d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)] + \frac{\mu d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)}{1 + d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)}$$

we obtain

$$\begin{aligned} \|d(f(\kappa), f(\vartheta_n))\| &\leq \lambda[\|d(\vartheta_n, \kappa) + d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)\|] \\ &\quad + \frac{\mu \|d(\vartheta_n, \varpi_n) d(f(\kappa), \kappa)\|}{\|1 + d(\vartheta_n, \varpi_n) + d(f(\kappa), \kappa)\|}, \end{aligned}$$

as $n \to \infty$, we get $d(f(\kappa), \kappa) = 0$. Hence $f(\kappa) = \kappa$. Therefore κ is a fixed point of f.

If ρ is another fixed point of f, then $f(\rho) = \rho$, $\rho \in G \cap H$, and hence,

$$\begin{aligned} d(\kappa, \rho) &= d(f(\kappa), f(\rho)) \precsim_{i_2} \lambda[d(\kappa, \rho) + d(\kappa, f(\kappa)) + d(f(\rho), \rho)] \\ &\quad + \frac{\mu d(\kappa, f(\kappa)) d(f(\rho), \rho)}{1 + d(\kappa, f(\kappa)) + d(f(\rho), \rho)}. \end{aligned}$$

Therefore $\|d(\kappa, \rho)\| = 0$ so that $\kappa = \rho$. So f has a UFP. □

4. Conclusions

All fixed point theorems in bicomplex valued bipolar metric spaces can be regarded as generalizations of fixed point theorems in bicomplex valued metric spaces which are generalization of complex valued metric spaces. Therefore, studies of fixed point results in bicomplex valued bipolar metric spaces are significant.

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