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Numerical solution of a coupled system of fractional order integro differential equations by an efficient numerical method based on the second kind Chebyshev polynomials

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Abstract. In this paper, an efficient numerical method based on operational matrices of the second kind Chebyshev polynomials is used for the solution of a coupled system of fractional order integro differential equations that the fractional derivative is given in Caputo's sense. The operational matrices of the second kind Chebyshev polynomials reduces the given equations to a system of linear algebraic equations. Approximate solution is calculated by extending the functions in terms of second kind Chebyshev polynomials and applying operational matrices. Unknown coefficients are obtained by solving final system of linear equations. Also convergence analysis and error bound of the solution are studied in this paper. Moreover, to check the reliability and accuracy of the given method. The numerical examples have been showed and the results of the described method are compared with the Haar wavelet method. The obtained results authenticate that the displayed method is effortless to analyze and perform such types of problems. All methods for the proposed method are applied in MATLAB ($R2020b$) software.

1. Introduction

The issue of factional calculus (FC) prepare generalized notion of classical calculus. In the last decades, FC has been utilized as a practical tool to model chemical processes, physical, signal processing, viscoelasticity, radiative equilibrium and diffusion processes that are discovered to be best described by fractional integro

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differential equations (FIDEs) [**15**, **2**, **9**, **17**, **16**]. FIDE plays an important and main role in many fields of engineering and scientific disciplines such as electric transmission, radiation, generalized voltage divider, economics and fluid mechanics [**14**, **27**]. It is worth mentioning that with the extension of FC, the action of many systems can be depicted by applying the FIDEs and fractional differential systems [**18**]. The analytical solutions developed for the FIDEs are very few and are limited to the solution of simple FIDEs, therefore the necessity of solving FIDEs by applying efficient and effective numerical methods accurately has emerged as an increasing interest and a key issue for several years in many research fields. Various researchers applied on the numerical solution of FIDEs. In some instances, if FIDEs do not have exact solution , then we are interested to discover their approximate solution. For example, Rawashdeh [**28**] presented a FIDEs using the Legendre wavelets method. Saleh et al. [**32**] proposed a Taylor expansion and Legendre wavelet method to solve FIDEs. Arikoglu and Okozol [**1**] employed the fractional differential transform method to solve the FIDEs. Mittal and Nigam [**23**] proposed the Adomian decomposition technique to solve the FIDEs. Vanani and Amnataei [**33**] used the operational Tau approximation method to solve the FIDEs. Yang and Hou [**39**] used the Laplace decomposition method to solve the FIDEs. Mohammed [**21**] employed the shifted Chebyshev polynomial and least squares methods to solve the FIDEs. Mahdy and Shwayyea [**22**] found the numerical solution of FIDEs by Laguerre polynomials pseudo spectral methods. Sweilam et al. [**31**] proposed a variational iteration method to solve FIDEs. Nazari and Shahmorad [**24**] employed the generalized differential transform method to solve the FIDEs. Xie et al. [**35**] proposed a numerical method based on the wavelet methods for solving a class of FIDEs. Pedas et al. [**26**] solved FIDEs numerically using a spline collocation method. Wang and Zhu [**34**] solved FIDEs numerically using a second kind Chebyshev wavelets method. Mokhtary [**20**] solved FIDEs numerically using an operational Jacobi Tau method. Biazar and Sadri [**4**] proposed numerical solution for the class of FIDEs using shifted Jacobi polynomials. The advantage of this new method with Chebychev polynomials consists mainly in the fact that it is easy to use and implement but gives good approximations of the solution at a given set of nodes. Also other authors apply numerical methods to obtain the approximate solutions of FIDEs, such as, Adomian decomposition technique [**10**], wavelet techniques [**30**], collocation technique [**12**, **13**], Block Pulse functions technique [**36**, **37**, **38**], variational iteration technique [**11**], combination of the parametric iteration method and the spectral collocation method [**7**], Haar wavelet technique [**3**], Lucas wavelets and the Legendre–Gauss quadrature rule [**8**] and fractional order operational matrix methods [**29**].

One of the numerical methods to obtain the approximate solutions of FIDEs is the orthogonal based methods and the main goal of applying orthogonal basis is that utilizing operational matrices given in terms of an orthogonal basis, the introduced equations transform into linear or nonlinear algebraic equations. In this paper, we show a spectral method based on the second kind Chebyshev polynomials to obtain the approximate solutions of a coupled system of FIDEs that are defined as:

$$
\begin{aligned}
&{}^{C}\mathbf{D}_t^{\mu} z(t) + z'(t) + \int_0^t w(\tau)d\tau = f(t),\\
&{}^{C}\mathbf{D}_t^{\nu} w(t) + w'(t) + \int_0^t z(\tau)d\tau = h(t),
\end{aligned}
\tag{1}
$$

subject to the following initial conditions:

$$
z(0) = z'(0) = w(0) = w'(0) = 0, \tag{2}
$$

where ${}^{C}\mathbf{D}_t^{\mu}$ and ${}^{C}\mathbf{D}_t^{\nu}$ show the Caputo fractional derivative operators of orders $\mu \in (1, 2]$ and $\nu \in (1, 2]$ with respect to t, $t \in [0, 1]$, respectively. Here $f(t)$, $h(t)$ are given known functions and $z(t)$, $w(t)$ are an unknown functions. In this paper, applying the spectral method based on operational matrices of the second kind Chebyshev polynomials (SKCPs) with the Riemann Liouville fractional integral operators respect to t, we decrease the coupled system of FIDEs to systems of algebraic equations. This method is easy and accurate to implement and performance in solving Eq. (1).

The outline of this paper is organized as follow. Section 2 gives some preliminaries and basic definitions of the fractional calculus. Moreover, we survey the SKCPs and show some of their properties. In Section 3, we describe a numerical method to solve a coupled system of FIDEs. Section 4 reports the convergence analysis of the system based on the SKCPs. In Section 5, some numerical examples are demonstrated to test the proposed method and compares the proposed method with the Haar wavelet method is given in [35] which show that the proposed method is accurate and efficient. Conclusion is shown in Section 6.

2. Mathematical preliminaries and definitions

In this section, we present some mathematical preliminaries and necessary definitions of the fractional calculus theory and main properties of the SKCPs that will be applied further in this paper.

Definition 2.1. The fractional integral operator of order $\alpha \geq 0$ for a function $u(t) \in L[0, 1]$ with respect to the variable t in the Riemann-Liouville type is given in [27, 14] as:

$$
\mathbf{I}_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) ds, \tag{3}
$$

where $\Gamma(\alpha)$ is the gamma function. Also, from the Riemann-Liouville fractional integral operator definition for $\alpha \geq 0$ and $\beta > -1$, we have:

$$\mathbf{I}_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}. \tag{4}$$

Definition 2.2. The fractional differential operator of a function $u(t) \in L[0,1]$ with respect to the variable t in Caputo sense is defined as:

$${}^{C}\mathbf{D}_t^{\alpha} u(t) = \mathbf{I}_t^{n-\alpha} \frac{d^n u(t)}{dt^n} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, & \alpha \in (n-1, n],\ n \in \mathbb{N}, \\ \frac{d^n u(t)}{dt^n}, & \alpha = n. \end{cases} \tag{5}$$

Some practical and useful properties of the of the Riemann-Liouville fractional integral operator and the Caputo fractional differential operator are presented below:

$$\begin{aligned} &{}^{C}\mathbf{D}_t^{\alpha} \mathbf{I}_t^{\alpha} u(t) = u(t), \\ &\mathbf{I}_t^{\alpha} {}^{C}\mathbf{D}_t^{\alpha} u(t) = u(t) - \sum_{j=0}^{n-1} u^{(j)}(0) \frac{t^j}{j!},\ \alpha \in (n-1, n],\ n \in \mathbb{N}. \end{aligned} \tag{6}$$

Definition 2.3. The well known SKCPs are given on the interval $[-1,1]$ and can be defined with the following recurrence relation:

$$\begin{aligned} &U_0 = 1,\ U_1(x) = 2x, \\ &U_j(x) = 2tU_{j-1}(x) - U_{j-2}(x),\ j = 2, 3, \ldots. \end{aligned} \tag{7}$$

In order to apply these polynomials on the interval $[0,T]$, we give the shifted SKCPs by considering the change of variable $x = \frac{2t}{T} - 1$. Then, shifted SKCPs are defined by:

$$\phi_j(t) = U_j(\frac{2t}{T} - 1),\ j = 0, 1, 2, \ldots, \tag{8}$$

where U_j is the SKCPs. The analytic model of the shifted SKCPs of degree j is defined by:

$$\phi_j(t) = \sum_{i=0}^{j} \frac{(-1)^{j-i}(j+i+1)! 2^{2j}}{(j-i)!(2j+1)! T^j} t^j. \tag{9}$$

The orthogonality property for the Shifted SKCPs with respect to the weight function $\omega(t)$ on $[0,T]$ can be expressed as:

$$\int_0^T \omega(t)\phi_m(t)\phi_n(t) = \begin{cases} \frac{T\pi}{4}, & n = m, \\ 0, & m \neq n, \end{cases} \tag{10}$$

where $\omega(t) = \sqrt{1 - (\frac{2t}{T} - 1)^2}$.

Theorem 2.1. [25] *The Riemann-Liouville fractional integral operator of of order α for the shifted SKCPs can be expanded in terms of the shifted SKCPs as follows:*

$$\boldsymbol{I}_t^{\alpha}\phi_j(t) \simeq \sum_{k=0}^{N} \Delta_{jk}^{\alpha}\phi_k(t), \tag{11}$$

where

$$\Delta_{jk}^{\alpha} = \frac{4(k+1)T^{\alpha}}{\sqrt{\pi}} \sum_{l=0}^{j} \frac{(-1)^{j-l}(j+l+1)!l!2^{2l}\Gamma(l+\alpha+\frac{3}{2})}{(j-l)!(2l+1)!\Gamma(j+l+\alpha+3)\Gamma(l+\alpha-k+1)}. \tag{12}$$

For a function $y(t) \in L[0,T]$, we may also expand its approximation by applying shifted SKCPs as:

$$y(t) = \sum_{k=0}^{\infty} a_k\phi_k(t), \tag{13}$$

where a_k is the coefficient and

$$a_k = \frac{4}{T\pi}\int_0^T \omega(\tau)y(\tau)\phi_k(\tau)d\tau,\ \ k = 0, 1, 2, \ldots. \tag{14}$$

Hence the function $y(t)$ can be expanded by truncating the infinite series given in Eq. (13) as follows:

$$y(t) \simeq \sum_{k=0}^{N} a_k\phi_k(t) = A^T\Phi(t), \tag{15}$$

where

$$\begin{aligned} A &= [a_0\ a_1\ \ \ldots\ \ a_N]^T, \\ \Phi(t) &= [\phi_0(t)\ \phi_1(t)\ \ \ldots\ \ \phi_N(t)]^T. \end{aligned} \tag{16}$$

3. Description of the suggested methodology for problem (1)

In this section, we describe the numerical method for the approximate solution of Eq. (1). To this aim, we approximate $\frac{d^2z}{\partial t^2}$ and $\frac{d^2w}{\partial t^2}$ applying the shifted second kind Chebyshev series as follows:

$$\frac{d^2z}{dt^2} = \sum_{i=0}^{N} z_i\phi_i(t) = Z^T\Phi(t), \tag{17}$$

$$\frac{d^2w}{dt^2} = \sum_{j=0}^{N} w_j\phi_j(t) = W^T\Phi(t), \tag{18}$$

where

$$Z = [z(t_0)\ z(t_1)\ \ldots\ z(t_N)]^T,$$
$$W = [w(t_0)\ w(t_1)\ \ldots\ w(t_N)]^T. \tag{19}$$

Integrating from Eqs. (17) and (18) with respect to the variable t, we obtain:

$$\frac{dz}{dt} = z'(0) + Z^T M\Phi(t), \tag{20}$$
$$\frac{dw}{dt} = w'(0) + W^T M\Phi(t), \tag{21}$$

where M is the $(N+1)\times(N+1)$ operational matrix of the first order integration of the Shifted SKCPs functions vector $\Phi(t)$ and

$$M = [\Delta^1_{ij}],\ \ i, j = 0, 1, \ldots, N. \tag{22}$$

Again, by integrating from Eqs. (20) and (21) with respect to the variable t and applying Eq. (2), we obtain:

$$z(t) = z(0) + Z^T M^2\Phi(t), \tag{23}$$
$$w(t) = w(0) + W^T M^2\Phi(t). \tag{24}$$

Now, by taking the Riemann-Liouville fractional integral operator of order μ to both sides of Eq. (1) and using Eq. (6), yields:

$$z(t) \ = \mathbf{I}^{\mu}_t f(t) - \mathbf{I}^{\mu}_t \left[\tfrac{dz}{dt} + \int_0^t w(\tau)d\tau\right], \tag{25}$$
$$w(t) \ = \mathbf{I}^{\nu}_t h(t) - \mathbf{I}^{\nu}_t \left[\tfrac{dw}{dt} + \int_0^t z(\tau)d\tau\right]. \tag{26}$$

The functions $f(t)$ and $h(t)$ in Eqs. (25) and (26) can be expanded in terms of the shifted second kind Chebyshev series as:

$$f(t) = \sum_{i=0}^{N} f_i\phi_i(t) = F^T\Phi(t), \tag{27}$$
$$h(t) = \sum_{i=0}^{N} h_i\phi_i(t) = H^T\Phi(t), \tag{28}$$

where

$$F = [f(t_0)\ f(t_1)\ \ldots\ f(t_N)]^T,$$
$$H = [h(t_0)\ h(t_1)\ \ldots\ h(t_N)]^T, \tag{29}$$

where $f(t_i)$ and $h(t_i)$ for $i = 0, 1, \ldots, N$ are known functions. To continue the method, the second part of the right hand side in Eqs. (25) and (26)can also be approximated in terms of the shifted second kind Chebyshev series applying Eqs. (22), (23) and (24) as follows:

$$\int_0^t z(\tau)d\tau = \int_0^t Z^T M^2 \Phi(\tau)d\tau = Z^T M^3 \Phi(t), \tag{30}$$

$$\int_0^t w(\tau)d\tau = \int_0^t W^T M^2 \Phi(\tau)d\tau = W^T M^3 \Phi(t). \tag{31}$$

Substituting Eqs. (20), (21) and (23)-(28), (30) and (31) into Eqs. (25) and (26) and using Eq. (12), we obtain:

$$Z^T M^2 \Phi(t) \ = F^T \mathbf{M}^{(\mu)} \Phi(t) - \left[Z^T M \mathbf{M}^{(\mu)} \Phi(t) + W^T M^3 \mathbf{M}^{(\mu)} \Phi(t) \right], \tag{32}$$

$$W^T M^2 \Phi(t) \ = H^T \mathbf{M}^{(\nu)} \Phi(t) - \left[W^T M \mathbf{M}^{(\nu)} \Phi(t) + Z^T M^3 \mathbf{M}^{(\nu)} \Phi(t) \right], \tag{33}$$

where $\mathbf{M}^{(\mu)}$ is the $(N+1) \times (N+1)$ operational matrix of the fractional order integration of order μ of the Shifted SKCPs functions vector $\Phi(t)$ and

$$\mathbf{M}^{(\mu)} = [\Delta_{ij}^{\mu}], \ i,j = 0,1,\ldots,N, \tag{34}$$
$$\mathbf{M}^{(\nu)} = [\Delta_{ij}^{\nu}], \ i,j = 0,1,\ldots,N.$$

Also, utilizing Eqs. (20), (21), (23) and (24) into the initial conditions considered in Eq. (2), yields:

$$Z^T M^2 \Phi(0) = Z^T M \Phi(0) = W^T M^2 \Phi(0) = W^T M \Phi(0) = 0, \tag{35}$$

In consequence, from Eqs. (32) and (33), we obtain:

$$\left(Z^T M^2 + Z^T M \mathbf{M}^{(\mu)} + W^T M^3 \mathbf{M}^{(\mu)} - F^T \mathbf{M}^{(\mu)} \right) \Phi(t) = 0, \tag{36}$$

$$\left(W^T M^2 + W^T M \mathbf{M}^{(\nu)} + Z^T M^3 \mathbf{M}^{(\nu)} - H^T \mathbf{M}^{(\nu)} \right) \Phi(t) = 0. \tag{37}$$

Finally, we have:

$$Z^T M^2 + Z^T M \mathbf{M}^{(\mu)} + W^T M^3 \mathbf{M}^{(\mu)} - F^T \mathbf{M}^{(\mu)} = 0, \tag{38}$$

$$W^T M^2 + W^T M \mathbf{M}^{(\nu)} + Z^T M^3 \mathbf{M}^{(\nu)} - H^T \mathbf{M}^{(\nu)} = 0. \tag{39}$$

To compute the unknown parameters we apply the collocation points $\frac{m}{N+2}T, \ m = 1,2,\ldots,N+1$. By solving Eqs. (38) and (39) associated with the initial conditions given in Eq. (35), we can find the unknown parameters Z, W and we can get the numerical solution of Eq. (1). Consequently, we can approximately obtain $z(t), w(t)$ applying Eqs. (23) and (24).

4. Convergence and error analysis

In this section, the convergence and error analysis of the derived numerical solution based on Shifted SKCPs are investigated. To do this purpose, we suppose that the determined functions in Eq. (1) provide the cases that the exact solutions $z(t), w(t)$ belongs to the Sobolev space $\mathbb{H}^{n+1}\Big((0,T)\Big)$, that $n \geq 0$. By applying properties of Sobolev spaces [6, 5], we gain the following results.

Theorem 4.1. [25] *Let $u(t) \in \mathbb{H}^{n+1}\Big((0,T)\Big)$ and $\mathbb{P}_N(u(t)) = \sum_{k=0}^{N} u_k\phi_k(t)$ be the truncated SKC series of the function $u(t)$. Then, we have:*

$$|\mathbb{E}_N(u)| = |u(t) - \mathbb{P}_N(u(t))| = O\big(N^{-n+1}\big), \tag{40}$$

when $N \to \infty$ for all $t \in (0,T)$.

Theorem 4.2. [19] *Let $u(t) \in [0,1]$ be N times continuously differentiable and $u_N(t) = \sum_{i=0}^{N} u_i\phi_i(t) = U^T\Phi(t)$ be its best square approximation function. Then we have:*

$$\|u(t) - u_N(t)\| \leq \frac{\lambda Q^{N+1}}{(N+1)!}\sqrt{\frac{\pi}{8}}, \tag{41}$$

where $\lambda = \max_{t\in[0,1]} u^{N+1}(t)$ and $Q = \max[t_0, 1-t_0]$.

4.1. Convergence analysis. In order to display the convergence analysis of Eq. (1), we consider the error functions $\mathbb{E}_N(z) = z(t) - z_N(t)$ and $\mathbb{E}_N(w) = w(t) - w_N(t)$, that $z(t), w(t)$ are the exact solutions of Eq. (1), and $z_N(t), w_N(t)$ are their approximation obtained by the proposed method, respectively. Then from Eqs. (25) and (26), we have:

$$z_N(t) \quad = \mathbf{I}_t^{\mu} f(t) - \mathbf{I}_t^{\mu}\left[\tfrac{dz_N}{dt} + \textstyle\int_0^t w_N(\tau)d\tau\right] + R_1(t), \tag{42}$$

$$w_N(t) \quad = \mathbf{I}_t^{\nu} h(t) - \mathbf{I}_t^{\nu}\left[\tfrac{dw_N}{dt} + \textstyle\int_0^t z_N(\tau)d\tau\right] + R_2(t). \tag{43}$$

where $R_1(t), R_2(t)$ are the residue terms and $f(t) = F^T\Phi(t),\ h(t) = H^T\Phi(t)$. Now, by subtracting Eqs. (25) and (26) from Eqs. (42) and (43), we get:

$$\begin{aligned} |R_1(t)| \quad &\leq |z(t) - z_N(t)| + |\mathbf{I}_t^{\mu} f(t) - \mathbf{I}_t^{\mu} F^T\Phi(t)| + |\mathbf{I}_t^{\mu} z'(t) - \mathbf{I}_t^{\mu} z_N'(t)| \\ &\quad + |\mathbf{I}_t^{\mu}\Big(\textstyle\int_0^t w(\tau)d\tau\Big) - \mathbf{I}_t^{\mu}\big(\textstyle\int_0^t w_N(\tau)d\tau\big)|, \end{aligned} \tag{44}$$

$$\begin{aligned} |R_2(t)| \quad &\leq |w(t) - w_N(t)| + |\mathbf{I}_t^{\mu} h(t) - \mathbf{I}_t^{\mu} H^T\Phi(t)| + |\mathbf{I}_t^{\mu} w'(t) - \mathbf{I}_t^{\mu} w_N'(t)| \\ &\quad + |\mathbf{I}_t^{\mu}\big(\textstyle\int_0^t z(\tau)d\tau\big) - \mathbf{I}_t^{\mu}\big(\textstyle\int_0^t z_N(\tau)d\tau\big)|. \end{aligned} \tag{45}$$

(46)

By applying Theorem 4.1, when $N \to \infty$, we gain the following approximations:

$$\begin{aligned}|z(t) - z_N(t)| &= O\big(N^{-n+1}\big),\\ |\mathbf{I}_t^{\mu} f(t) - \mathbf{I}_t^{\mu} F^T \Phi(t)| &= O\big(N^{-n+1}\big),\\ |\mathbf{I}_t^{\mu} z'(t) - \mathbf{I}_t^{\mu} z'_N(t)| &= O\big(N^{-n+1}\big),\\ |\mathbf{I}_t^{\mu}\Big(\int_0^t w(\tau)d\tau\Big) - \mathbf{I}_t^{\mu}\big(\int_0^t w_N(\tau)d\tau\big)| &= O\big(N^{-n+1}\big).\end{aligned} \tag{47}$$

Therefore

$$|R_1(t)| \to 0. \tag{48}$$

Similar to above relation, we get:

$$|R_2(t)| \to 0. \tag{49}$$

5. Numerical examples

This Section illustrates several numerical examples to present the effectiveness, practicability and accuracy of the proposed Shifted SKCPs method. In order to show the error of the method we consider the following notation:

$$\begin{aligned}\|\mathbb{E}_N(z)\|_2 &= \Big(\int_0^T \omega(\tau)\mathbb{E}_N^2(z)(\tau)d\tau\Big)^{\frac{1}{2}},\\ \|\mathbb{E}_N(w)\|_2 &= \Big(\int_0^T \omega(\tau)\mathbb{E}_N^2(w)(\tau)d\tau\Big)^{\frac{1}{2}}.\end{aligned} \tag{50}$$

Here we consider $T = 1$. All the numerical computations are carried out with the mathematical software MATLAB(R2020b). Also comparison of the absolute error of the computed numerical solution by the proposed method with the Haar wavelet method is introduced in [35], is illustrated.

TABLE 1. Absolute error $\mathbb{E}_N(z)$ for various values N for Example 5.1.

t	$N = 8$	$N = 16$	$N = 32$	$N = 64$
	Our method			
$\frac{1}{8}$	$1.541210e-22$	$2.865625e-23$	$4.695556e-24$	$6.036376e-26$
$\frac{2}{8}$	$6.164843e-22$	$1.146250e-22$	$1.878222e-23$	$2.414550e-25$
$\frac{3}{8}$	$1.387089e-21$	$2.579062e-22$	$4.226000e-23$	$5.432739e-25$
$\frac{4}{8}$	$2.465937e-21$	$4.585000e-22$	$7.512890e-23$	$9.658203e-25$
$\frac{5}{8}$	$3.853027e-21$	$7.164062e-22$	$1.173889e-22$	$1.509094e-24$
$\frac{6}{8}$	$5.548359e-21$	$1.031625e-21$	$1.690400e-22$	$2.173095e-24$
$\frac{7}{8}$	$7.551933e-21$	$1.404156e-21$	$2.300822e-22$	$2.957824e-24$

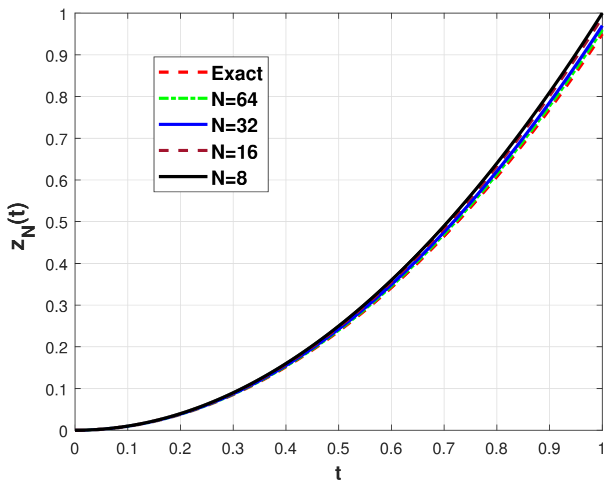

FIGURE 1. Numerical and exact solutions of the Example 5.1 for different values of N.

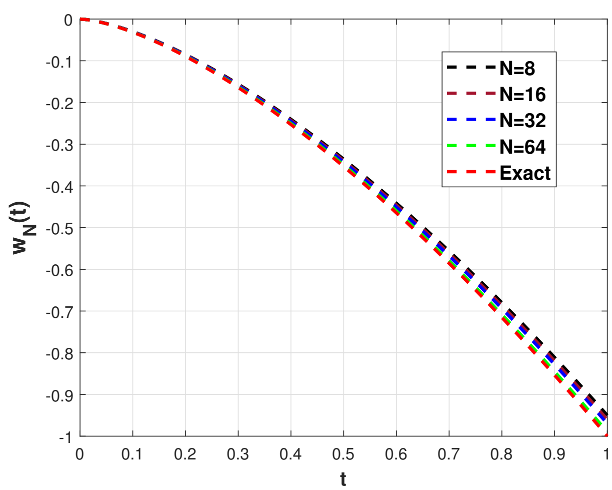

FIGURE 2. Numerical and exact solutions of the Example 5.1 for different values of N.

Example 5.1. Consider the following FIDEs as [35]:

$$\begin{aligned} {}^{C}\mathbf{D}_t^{1.5} z(t) + z'(t) + \int_0^t w(\tau)d\tau &= f(t), \\ {}^{C}\mathbf{D}_t^{1.25} w(t) + w'(t) + \int_0^t z(\tau)d\tau &= h(t), \end{aligned} \tag{51}$$

with the following initial conditions:

$$z(0) = z'(0) = 0, \;\; w(0) = w'(0) = 0, \tag{52}$$

TABLE 2. Absolute error $\mathbb{E}_N(w)$ for various values N for Example 5.1.

t	$N=8$	$N=16$	$N=32$	$N=64$
	Our method			
$\frac{1}{8}$	$4.359202e-22$	$8.105211e-23$	$1.328103e-23$	$1.707345e-25$
$\frac{2}{8}$	$1.232968e-21$	$2.292500e-22$	$3.756445e-23$	$4.829101e-25$
$\frac{3}{8}$	$2.265108e-21$	$4.211591e-22$	$6.901030e-23$	$8.871626e-25$
$\frac{4}{8}$	$3.487362e-21$	$6.484169e-22$	$1.062483e-22$	$1.365876e-24$
$\frac{5}{8}$	$4.873736e-21$	$9.061901e-22$	$1.484865e-22$	$1.908869e-24$
$\frac{6}{8}$	$6.406693e-21$	$1.191217e-21$	$1.951906e-22$	$2.509274e-24$
$\frac{7}{8}$	$8.073356e-21$	$1.501106e-21$	$2.459682e-22$	$3.162047e-24$

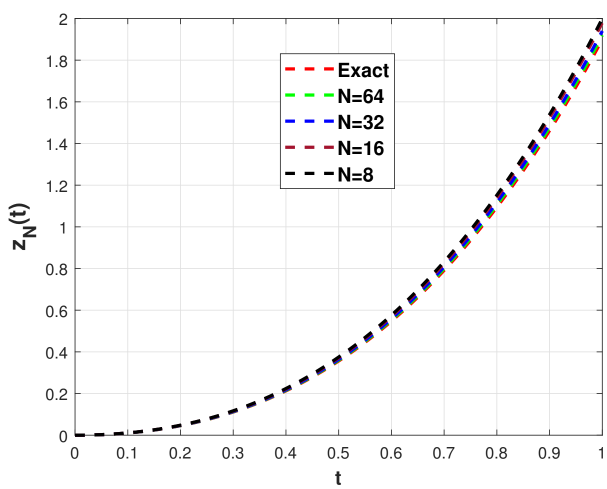

FIGURE 3. Numerical and exact solutions of the Example 5.2 for different values of N.

where

$$\begin{aligned} f(t) &= \tfrac{4}{\sqrt{\pi}}t^{0.5} + 2t - \tfrac{2}{5}t^{2.5}, \\ h(t) &= \tfrac{\Gamma(2.5)}{\Gamma(1.25)}t^{0.25} - 1.5t^{0.5} + \tfrac{1}{3}t^3. \end{aligned} \tag{53}$$

The exact solutions of this system are $z(t) = t^2$ and $w(t) = -t^{1.5}$. We have applied the proposed method to this system and demonstrated graphically the numerical results between the numerical solutions proposed by the presented method and the exact solutions of Eq. (51) with $N = 8, 16, 32, 64$ in Figs. 1 and 2. The authors of [35] applied Haar wavelet method to solve this system. The absolute error obtained by using the our method with $N = 8, 16, 32, 64$ are reported in Tables 1 and 2. From Tables 1 and 2, we see that the proposed method is an effective and practicable tool for solving this system.

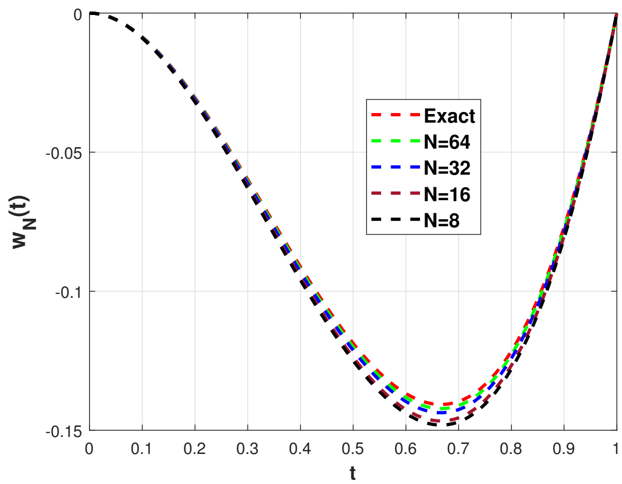

FIGURE 4. Numerical and exact solutions of the Example 5.2 for different values of N.

TABLE 3. Comparison of absolute error $\mathbb{E}_N(z)$ with various values of N for Example 5.2.

t	$N=8$	$N=16$	$N=32$	$N=64$
	Our method			
$\frac{1}{8}$	$1.733862e-22$	$3.223828e-23$	$5.282501e-24$	$6.790924e-26$
$\frac{2}{8}$	$7.706054e-22$	$1.432812e-22$	$2.347778e-23$	$3.018188e-25$
$\frac{3}{8}$	$1.907248e-21$	$3.546210e-22$	$5.810751e-23$	$7.470016e-25$
$\frac{4}{8}$	$3.698906e-21$	$6.877500e-22$	$1.126933e-22$	$1.448730e-24$
$\frac{5}{8}$	$6.261169e-21$	$1.164160e-21$	$1.907569e-22$	$2.452278e-24$
$\frac{6}{8}$	$9.709628e-21$	$1.805343e-21$	$2.958200e-22$	$3.802917e-24$
$\frac{7}{8}$	$1.415987e-20$	$2.632792e-21$	$4.314042e-22$	$5.545921e-24$
	Haar wavelet method [35]			
$\frac{1}{8}$	$2.371827e-5$	$9.372818e-7$	$4.231908e-7$	$7.902391e-9$
$\frac{2}{8}$	$4.378379e-5$	$3.476292e-6$	$5.373637e-7$	$9.379809e-9$
$\frac{3}{8}$	$6.261838e-5$	$5.371987e-6$	$6.749280e-7$	$3.293890e-8$
$\frac{4}{8}$	$7.049200e-5$	$6.372830e-6$	$8.318309e-7$	$4.238209e-8$
$\frac{5}{8}$	$8.381098e-5$	$8.381903e-6$	$2.447810e-6$	$6.351738e-8$
$\frac{6}{8}$	$2.361873e-4$	$9.748292e-6$	$3.183190e-6$	$7.487310e-8$
$\frac{7}{8}$	$3.012919e-4$	$1.371298e-5$	$3.489318e-6$	$8.127288e-8$

TABLE 4. Comparison of absolute error $\mathbb{E}_N(w)$ with various values of N for Example 5.2.

t	$N=8$	$N=16$	$N=32$	$N=64$
	Our method			
$\frac{1}{8}$	$1.348559e-22$	$2.507421e-23$	$4.108612e-24$	$5.281829e-26$
$\frac{2}{8}$	$4.623632e-22$	$8.596875e-23$	$1.408666e-23$	$1.810913e-25$
$\frac{3}{8}$	$8.669311e-22$	$1.611914e-22$	$2.641250e-23$	$3.395462e-25$
$\frac{4}{8}$	$1.232968e-21$	$2.292500e-22$	$3.756445e-23$	$4.829101e-25$
$\frac{5}{8}$	$1.444885e-21$	$2.686523e-22$	$4.402084e-23$	$5.659103e-25$
$\frac{6}{8}$	$1.387089e-21$	$2.579062e-22$	$4.226000e-23$	$5.432739e-25$
$\frac{7}{8}$	$9.439916e-22$	$1.755195e-22$	$2.876028e-23$	$3.697280e-25$
	Haar wavelet method [35]			
$\frac{1}{8}$	$3.472093e-5$	$9.029109e-7$	$3.039091e-7$	$3.480938e-8$
$\frac{2}{8}$	$5.312731e-5$	$3.319820e-6$	$5.380936e-7$	$5.361870e-8$
$\frac{3}{8}$	$5.938319e-5$	$4.381098e-6$	$6.472987e-7$	$6.371091e-8$
$\frac{4}{8}$	$6.738170e-5$	$6.419302e-6$	$7.371873e-7$	$8.381038e-8$
$\frac{5}{8}$	$7.381092e-5$	$7.981023e-6$	$9.463619e-7$	$2.479837e-7$
$\frac{6}{8}$	$8.849200e-5$	$8.251735e-6$	$1.237297e-6$	$4.328102e-7$
$\frac{7}{8}$	$2.371980e-4$	$9.498249e-6$	$2.383010e-6$	$5.381098e-7$

TABLE 5. Comparison of 2-norm errors $\|\mathbb{E}_N(z)\|_2$ and $\|\mathbb{E}_N(w)\|_2$ with various values of N for Example 5.2.

	$N=16$	$N=32$	$N=64$
	Our method		
$\|\mathbb{E}_N(z)\|_2$	$3.9326e-12$	$1.5919e-12$	$1.8049e-13$
$\|\mathbb{E}_N(w)\|_2$	$1.6520e-12$	$6.6873e-13$	$7.5822e-14$
	Haar wavelet method [35]		
	$3.478023e-4$	$6.371379e-5$	$9.371983e-7$
	$5.371897e-4$	$8.381098e-5$	$2.387639e-6$

Example 5.2. Consider the following FIDEs as [35]:

$$
\begin{aligned}
{}^{C}\mathbf{D}_t^{1.75} z(t) + z'(t) + \int_0^t w(\tau)d\tau &= f(t),\\
{}^{C}\mathbf{D}_t^{1.75} w(t) + w'(t) + \int_0^t z(\tau)d\tau &= h(t),
\end{aligned}
\tag{54}
$$

subject to the following initial conditions:

$$
z(0) = z'(0) = 0, \;\; w(0) = w'(0) = 0, \tag{55}
$$

where

$$\begin{aligned} f(t) &= \tfrac{6}{\Gamma(2.25)}t^{1.25} + 3t^2 + 2t + \tfrac{t^3(3t-4)}{12}, \\ h(t) &= \tfrac{6}{\Gamma(2.25)}t^{1.25} + 3t^2 - 2t + \tfrac{t^3(3t+4)}{12}. \end{aligned} \tag{56}$$

The exact solutions of Eq. (54) subject to the given initial conditions are $z(t) = t^2(t+1)$ and $w(t) = t^2(t-1)$. We have used the proposed method to this system and illustrated a comparison between the numerical solutions proposed by the method to the exact solutions of Eq. (54) with $N = 8, 16, 32, 64$ in Figs. 3 and 4. Tables 3 and 4 compares the absolute errors between the exact and numerical solutions given by both the our method and the Haar wavelet method [**35**] to this system, with $N = 8, 16, 32, 64$. Due to Tables 3 and 4, the our method show more accurate solutions by applying only a small number of the basis functions. Also, Tables 3, 4 and 5 present that our method can get a higher convergence result when $N = 64$.

6. Conclusion

In this paper, a numerical method based on the shifted SKCPs and their operational matrix of fractional integration has been demonstrated for approximate solutions of a coupled system of FIDEs. The fractional order derivative is given in the Caputo sense. By using this operational matrix and properties of the shifted SKCPs we reduced the main coupled system to the coupled system of solving a system of algebraic systems. By solving the linear systems, approximate solutions are calculated. The convergence analysis of the method was widely investigated. Finally several examples are given to demonstrate the practical efficiency and applicability of the described method for solving coupled system of FIDEs and the obtained numerical results were compared with exact solutions. The outcomes of the present method have introduced very good accuracy over the cited work [**35**]. Matlab was applied for computations in this paper.

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