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On a nonlinear abstract second order integrodifferential equation-part I

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Abstract. The object of this paper is to study the existence, uniqueness and continuation of the solutions of nonlinear second order integrodifferential equations.

The theory of infinitesimal generator of C_0- semigroup in a Banach space, the fixed point theorems of Schauder and Banach are used to establish our main results.

1. Introduction

Consider the nonlinear second order integrodifferential equation of the form

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t)) + \int_{t_0}^{t} [a(t,s)g_0(s,u(s)) + g_1(t,s,u(s))]ds, \quad (1)$$

$$u(t_0) = \phi, u'(t_0) = \psi, t \geq t_0 \geq 0$$

where A and B are linear (ingeneral unbounded) operators in Banach space X.
The theory of existence, uniqueness and other properties of the solutions of ordinary higher order differential equations are extensively developed during the past few years but very little attention is given to the abstract nonlinear higher order differential and integrodifferential equation.Travis and Webb ([10],[11],[12]) have studied the special form of (1) when operator A=0, by using the theory of strongly continuous cosine family C (t), $t \, \epsilon \, R$ of bounded linear operators in a Banach space X. Sandefur [9] and Aviles and Sandefur [1] have studied the special form of (1)

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when integral on the right side is absent by using the factorization method coupled with the successive approximation technique. The equation (1) serves as an abstract formation of many partial integrodifferential equations which arise in the wave equations (possibly damped or strongly damped), the telegraph equation, the vibrating beam and other physical phenomenon (see [1],[2],[3]). The purpose of this paper is to study the existence, uniqueness and continuation of the solutions of (1) by using the method of factorization introduced in [1],[5],[9]. Examples are given to the special cases to illustrate the usefulness of our results to study the physical models (for example see [10],[11],[12]). The main tools employed in our analysis are based on the application of Schauder and Banach fixed point theorems.

2. Statement of the results

Before we state our results, we give the following basic concepts and definitions used in our subsequent discussion.
Let X be a Banach space with the norm $||.||$ and A_j , $j = 1, 2$, is infinitesimal generator of C_0–semi group $T_j(t)$, $j = 1, 2$ and $t \geq 0$ on a Banach space. The set of bounded linear operators $\{T(t), t\epsilon R_+\}$, $R_+ = [0, \infty)$, is C_0–semigroup on X if

(1) $T(t + s) = T(t)T(s) = T(s).T(t), t, s \geq 0$,
(2) $T(0) = I$ (the identity operator)
(3) $T(.)$ is strongly continuous in $t\epsilon R_+$,
(4) $||T(t)|| \leq Me^{\omega t}$, for some $M > 0$ and ω, $t\epsilon R_+$ ([See [6], p. 276]).

We assume the existence of linear operators A_j , $j = 1, 2$ (not necessarily commute) generates C_0 – semi group $T_j(t), j = 1, 2$. Then equation (1) can be written as

$$\begin{aligned} u''(t) - (A_1 + A_2)u'(t) + A_2A_1u(t) \\ = f(t, u(t)) + \int_{t_0}^{t} [a(t, s)g_0(s, u(s)) + g_1(t, s, u(s))]ds, \end{aligned} \tag{2}$$

$$u(t_0) = \phi, u'(t_0) = \psi, t \geq t_0.$$

A continuous solution u of the integral equation

$$\begin{aligned} u(t) = T(t - t_0)\phi + \int_{t_0}^{t} T_1(t - \tau)T_2(\tau)(\psi - A_1\phi)d\tau \\ + \int_{t_0}^{t}\int_{t_0}^{\tau} T_1(t - \tau)T_2(\tau - s)\int_{t_0}^{s}[a(s, \eta)g_0(\eta, u(\eta)) \\ + g_1(s, \eta, u(\eta))]d\eta ds d\tau \\ + \int_{t_0}^{t}\int_{t_0}^{s} T_1(t_1 - \tau)T_2(\tau - s)f(s, u(s))ds d\tau, \end{aligned} \tag{3}$$

where $\phi\epsilon D(A_1)$, is called the mild solution of (2). A C_0-semi group T(t) is called compact for $t > t_0 \geq 0$ if for every $t > t_0 \geq 0$, T(t)is compact operator. $T(t)$ is called compact if it is compact for $t > 0$.
The motivation for this form of a mild solution came from studying (1) in the factored form.

$$((d/dt) - A_1)((d/dt) - A_2)u = f(t, u(t)) + \int_{t_0}^{t} [a(t.s)g_0(t, u(t)) + g_1(t, s, u)]ds \quad (4)$$

Suppose A_1 and A_2 also commute, the commuting means that $(\lambda_1 I - A_1)^{(-1)}$ and $(\lambda_2 I - A_2)^{(-1)}$ commute for all λ_j in resolvent set of $A_j, j = 1, 2$. In this case we could also say that u is a mild solution of (2) if it satisfies

$$\begin{aligned} u(t) = \frac{1}{2}[(T_1(t-t_0) + T_2(t-t_0))\phi + \int_{t_0}^{t} T_1(t-\tau)T_2(\tau)(\psi - A_1\phi)d\tau \\ + \int_{t_0}^{t} T_2(t-\tau)T_1(\tau)(\psi - A_2\phi)d\tau \\ + \int_{t_0}^{t}\int_{t_0}^{\tau} [T_1(t-\tau)T_2(\tau-s) + T_2(t-\tau)T_1(\tau-s)]\{f(s, u(s)) \\ + \int_{t_0}^{s} [a(s,\eta)g_0(\eta, u(\eta)) + g_1(s,\eta,u(\eta))]d\eta\}dsd\tau] \quad (5) \end{aligned}$$

where $\phi\epsilon D(A_1) \cap D(A_2)$. This is an average of (3) by interchanging the roles of T_1 and T_2. The advantage of (5) is expected symmetry of T_1 and T_2.
For convenience, we list the following hypotheses used in our subsequent discussion.

HYPOTHESIS 1. (H_1) Let $f, g_0 : [0, \alpha) \times U \to X$ and $g_1 : [0, \alpha) \times [0, \alpha) \times U \to X, \alpha > 0$, be continuous functions where U is an open subset of X. Let $a : [0, \alpha) \times [0, \alpha) \to R$ be continuous and satisfies the uniform Hölder's continuity condition in the first and second arguments with the exponent ρ i.e. there exists a positive constant b_0 such that

$$|a(t_1, s_1) - a(t_2, s_2)| \leq b_0(|t_1 - t_2|^\rho + |s_1 - s_2|^\rho),$$

for all $t_1, t_2, s_1, s_2\epsilon[0, \alpha)$.

HYPOTHESIS 2. (H_2) Let $f, g_0 : R_+ \times X \to X$ and $g_1 : R_+ \times R_+ \times X \to X$ be continuous functions. Let $a : R_+ \times R_+ \to R$ be uniform Hölder continuous with $0 < \rho \leq 1$ and constant b_0 as defined in (H_1).

We need the following Lemmas in our subsequent discussion;

Lemma 2.1. *(Goldstein [See [4], p. 49]) If $g\epsilon C'[R_+, X]$ and*

$$w(t) = \int_{0}^{t} T(t-s)g(s)ds, \quad (6)$$

where T(.) is a semi-group generated by a linear operator A on X, then

$$w \epsilon C(R_+, D(A)) \cap C'(R_+, X)$$

and

$$w'(t) = T(t)g(0) + \int_0^t T(t-s)g'(s)ds, \tag{7}$$

and

$$Aw'(t) = T(t)g(0) - g(0) + \int_0^t [-g(s) + T(t-s)g'(s)]ds. \tag{8}$$

Lemma 2.2. *(Murge and Pachpatte [See [7], p. 28])*
Let X be a Banach space. Let F be an operator which maps the elements of X into itself for which F^r is a contraction, where r is a positive integer. Then F has a unique fixed point in X.

Lemma 2.3. *(Pazy [See [8],p. 49]) Let T(t) be C_0– semi group. If T(t) is compact for $t > t_0$ then T(t) is continuous in uniform operator topology for $t > t_0$.*

We are now in a position to state our results to be proved in this paper.

Theorem 2.4. *Let the hypothesis (1) be satisfied and $-A_i, i = 1, 2$ be infinitesimal generator of a compact semigroup $T_i(t), i = 1, 2$. Then for every $\phi \epsilon D(A_1) \subset U$ and $\psi \epsilon X$ there exists a $t_1(\phi, \psi)$; $0 < t_1 < \alpha$, such that the initial value problem (2) when $t_0 = 0$ has a mild solution $u \epsilon C([0, t_1], U)$ i.e. u(t) is a continous function from [0,t_1] to U.*

Theorem 2.5. *Let the hypothesis (2) be satisfied and $-A_i, i = 1, 2$ be infinitesimal generator of a compact semi-group, $T_i(t), t \geq 0, i = 1, 2$ on X. Then for every $\phi \epsilon D(A_1)$ the initial value problem (2) when $t_0 = 0$ has a mild solution u on the maximal interval of existence $[0, t_{max})$, if $t_{max} < \infty$, then*

$$\lim_{t \uparrow t_{max}} ||u(t)|| = \infty. \tag{9}$$

Remark 2.6. *It is to be noted that Sandefur in [9], Avile and Sandefur in [1], have studied the existence, uniqueness, continuity and other properties of (1) when*

$$\int_{t_0}^t [a(t,s)g_0(s, u(s)) + g_1(t, s, u(s))]ds = 0,$$

and $t_0 = 0$ by the method of successive approximation our method and conditions on nonlinear functions involving in (1) are different from those used in ([1],[9]).

Remark 2.7. *If $A_1 = -A_2$ then $T_2(t) = T(-t)$ and if we define*

$$C(t) = \frac{(T(t) + T(-t))}{2}, f(t, u(t)) = f(t) \text{ and } g(t, u(t)) = 0,$$

then (3) reduces to semi linear wave equation studied by Travis and Webb ([10],[11]), by using the theory of strongly continuous Cosine family $C(t)$ for $t\epsilon R$ of bounded linear operators in the Banach space X.

However our technique and assumptions on the functions f, g_0 and g_1 are different from those of [10], [11].

3. Proofs of Theorem 1 and 2

Proof. Our interest here is only in local solutions, we can assume that $\alpha < \infty$. Let $||T_i(t)|| \le M_i, i = 1, 2$, for $0 \le t \le \alpha$ and $t > 0, \rho > 0$ be such that

$B_{\rho_1}(\phi) = \{\nu : ||\nu - \phi|| \le \rho_1\} \subset U; ||f(s,\nu)|| \le N_1; ||g_0(s,\nu)|| \le N_2; ||g_1(t,s,\nu)|| \le N_3,$

for $\nu \epsilon B_{\rho_1}(\phi)$ and $M_3 = max|a(t,s)|, 0 \le s \le t \le t'$. Choose $t'' > 0$, such that

$$||T_1(t)\phi - \phi|| < \rho_1/2,$$

for $0 \le t \le t''$ and let

$$t_1 = \min\{t', t'', \alpha, \frac{\rho_1}{2M_1M_2[||\psi - A_1\phi|| + \{(M_3N_2 + N_3)\alpha + N_1\}\alpha]}\}.$$

Set

$$Y = C([0, t_1]; X)$$

and

$$Y_0 = \{u : u\epsilon Y; u(0) = \phi, u(t)\epsilon B_{\rho_1}(\phi)\},$$

Y_0 is clearly bounded closed convex subset of Y. We define a mapping $F : Y \to Y_0$, by

$$\begin{aligned}(Fu)(t) &= T_1(t)\phi + \int_0^t T_1(t-\tau)T_2(\tau)(\psi - A_1\phi)d\tau \\ &+ \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau - s)\int_0^s [a(s,\eta)g_0(\eta, u(\eta)) + g_1(s,\eta,u(\eta))]d\eta ds d\tau \\ &+ \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau - s)f(s,u(s))dsd\tau. \quad (10)\end{aligned}$$

From (10) and using assumptions on f, g_0, g_1, a and $T_i, i = 1, 2$, we obtain,

$$\begin{aligned}||(Fu)(t) - \phi|| &\le ||T_1(t)\phi - \phi|| + \int_0^t ||T_1(t-\tau)||||T_2(\tau)||||\psi - A_1\phi||d\tau \\ &+ \int_0^t \int_0^\tau ||T_1(t-\tau)||||T_2(\tau - s)|| \int_0^s |a(s,\eta)|||g_0(\eta, u(\eta))||d\eta ds d\tau \\ &+ \int_0^t \int_0^\tau ||T_1(t-\tau)||||T_2(\tau - s)|| \int_0^s ||g_1(s,\eta,u(\eta))||d\eta ds d\tau \\ &+ \int_0^t \int_0^\tau ||T_1(t-\tau)||||T_2(\tau - s)||||f(s,u(s))||dsd\tau \quad (11)\end{aligned}$$

$$\leq \frac{\rho_1}{2} + M_1 M_2 t[||\psi - A_1\phi|| + (M_3 N_2 + N_3)\frac{t^2}{3!} + N_1 \frac{t}{2!}]$$
$$\leq \frac{\rho_1}{2} + M_1 M_2 t_1[||\psi - A_1\phi|| + \{(M_3 N_2 + N_3)\alpha + N_1\}\alpha]$$
$$\leq \frac{\rho_1}{2} + \frac{\rho_1}{2}$$
$$\leq \rho_1$$

This shows that F maps into Y_0 into Y_0. The continuity of F follows from the continuity of f, g_0 and g_1. Moreover, F maps Y_0 into precompact subsets of Y_0.
To prove this, we first show that for every fixed $t, 0 \leq t \leq t_1$.
The set

$$Y_0(t) = \{(Fu)(t) : u\epsilon Y_0\}$$

is precompact in X. For $t = 0$, it is obvious since $Y_0(0) = \{\phi\}$. Let $t > 0$ be fixed. For $0 < \epsilon < t$, set

$$(F_\epsilon u)(t) = T_1(t)\phi + \int_0^{t-\epsilon} T_1(t-\tau)T(\tau)(\psi - A_1\phi)d\tau$$
$$+ \int_0^{t-\epsilon}\int_0^{\tau} T_1(t-\tau)T_2(\tau - s)\int_0^s a(s,\eta)g_0(\eta, u(\eta))d\eta ds d\tau$$
$$+ \int_0^{t-\epsilon}\int_0^{\tau} T_1(t-\tau)T_2(\tau - s)\int_0^s g_1(s,\eta, u(\eta))d\eta ds d\tau$$
$$+ \int_0^{t-\epsilon}\int_0^{\tau} T_1(t-\tau)T_2(\tau - s)f(s, u(s))ds d\tau \quad (12)$$

$$= T_1(t)\phi + T_1(\epsilon)\int_0^{t-\epsilon} T_1(t-\tau-\epsilon)T_2(\psi - A_1\phi)d\tau$$
$$+ T_1(\epsilon)\int_0^{t-\epsilon} T_1(t-\tau-\epsilon)T_2(\tau - s)\int_0^s a(s,\eta)g_0(\eta, u(\eta))d\eta ds d\tau$$
$$+ T_1(\epsilon)\int_0^{t-\epsilon}\int_0^{\tau} T_1(t-\tau-\epsilon)T_2(\tau - s)\int_0^s g_1(s,\eta, u(\eta))d\eta ds d\tau$$
$$+ T_1(\epsilon)\int_0^{t-\epsilon}\int_0^{\tau} T_1(t-\tau-\epsilon)T_2(\tau - s)f(s, u(s))ds d\tau$$

Since $T_i(t), i = 1, 2,$ are compact for every $t > 0$, the set

$$Y_\epsilon = \{(F_\epsilon u)(t) : u\epsilon Y_0\},$$

is precompact in X for every $\epsilon, 0 < \epsilon < t$. Furthermore, for $u \epsilon Y_0$, from (10), (12) and conditions on f, g_0, g_1, a and $T_i, i = 1, 2$. We obtain,

$$\begin{aligned} ||(Fu)(t) - (F_\epsilon u)(t)|| \leq & \int_{t-\epsilon}^{t} ||T_1(t-\tau)|| ||T_2(\tau)|| ||\psi - A_1\phi|| d\tau \\ & + \int_{t-\epsilon}^{t}\int_0^{\tau} ||T_1(t-\tau)|| ||T_2(\tau - s)|| \int_0^s [||a(s,\eta)|| ||g_0(\eta, u(\eta))|| \\ & + ||g_1(s,\eta,u(\eta))||] d\eta ds d\tau \\ & + \int_{t-\tau}^{t}\int_0^{\tau} ||T_1(t-\tau)|| ||T_2(\tau - s)|| || ||f(s,u(s))|| ds d\tau \qquad (13) \end{aligned}$$

$$\leq M_1 M_2 \epsilon [||\psi - A_1\phi|| + \frac{M_3 N_2 + N_3}{3!}\{\alpha^2 + (\alpha - \epsilon)(2\alpha - \epsilon)\} + \frac{N_1}{2!}\{2\alpha - \epsilon\}]$$

which implies that $Y_0(t)$ is totally bounded i.e. precompact in X. We continue to show that

$$F(Y_0) = \bar{Y} = \{Fu : u \epsilon Y_0\},$$

is an equicontinuous family of functions. Let $t_2 > t_1 > 0$. Using (10) and conditions on f, g_0 and g_1, we get

$$\begin{aligned} ||(Fu)(t_1) - (Fu)(t_2)|| \leq & ||(T_1(t_1) - T_1(t_2))\phi|| \\ & + \int_0^{t} ||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||T_2(\tau)|| ||\psi - A_1\phi|| d\tau \\ & + \int_{t_1}^{t_2} ||T_1(t_2-\tau)|| ||T_2(\tau)|| ||\psi - A_1\phi|| d\tau \\ & + \int_0^{t_1}\int_0^{\tau} [||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||T_2(\tau - s)|| \int_0^s [||a(s,\eta)|| ||g_0(\eta, u(\eta))|| \\ & + ||g_1(s,\eta,u(\eta))||]] d\eta ds d\tau \\ & + \int_{t_1}^{t_2}\int_0^{\tau} ||T_1(t_2-\tau)|| ||T_2(\tau - s)|| \int_0^s [||a(s,\eta)|| ||g_0(\eta, u(\eta))|| \\ & + ||g_1(s,\eta,u(\eta))||] d\eta ds d\tau \\ & + \int_0^{t_1}\int_0^{\tau} ||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||T_2(\tau - s)|| ||f(s,u(s))|| ds d\tau \\ & + \int_{t_1}^{t_2}\int_0^{\tau} ||T_1(t_2-\tau)|| ||T_2(\tau - s)|| ||f(s,u(s))|| ds d\tau \qquad (14) \end{aligned}$$

$$\leq ||(T_1(t_1) - T_1(t_2))\phi||$$
$$+ \int_0^{t_1} ||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||T_2(\tau)|| ||\psi - A_1\phi|| d\tau$$
$$+ M_1 M_2 ||\psi - A_1\phi|| (t_2 - t_1)$$
$$+ \int_0^{t_1}\int_0^{\tau} ||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||T_2(\tau - s)|| \int_0^s [M_3N_2 + N_3] d\eta ds d\tau$$
$$+ M_1M_2 \frac{(M_3N_2+N_3)}{3!}(t_2^3 - t_1^3)$$
$$+ \int_0^{t_1}\int_0^{\tau} ||T_1(t_1-\tau) - T_2(\tau - s)|| ||T_2(\tau - s)|| N_1 ds d\tau$$
$$+ M_1M_2 \frac{N_1}{2!}(t_2^2 - t_1^2)$$
$$\leq ||(T_1(t_1) - T_1(t_2))\phi|| + M_2 \int_0^{t_1} [||T_1(t_1-\tau) - T_1(t_2-\tau)|| ||\psi - A_1\phi||$$
$$+ \int_0^{\tau} \{N_1 + (M_3N_2+N_3)\int_0^s d\eta\} ds] d\tau + M_1M_2(t_2-t_1)[||\psi - A_1\phi||$$
$$+ \frac{(M_3N_2+N_3)}{3!}(t_2^2 + t_1^2 + t_1t_2) + \frac{N_1}{2!}(t_1+t_2)]$$

The right-hand side of the above inequality is free from $u\epsilon Y_0$ and approaches to zero as $t_2 - t_1 \to 0$. By Lemma 2.3, compactness of $T_i(t), i = 1,2, t > 0$, gives the continuity of $T_i(t), i = 1,2$ in the uniform topology. It is also clear that $\bar{Y}$ is bounded in Y. Now, by Arzela-Ascoli's theorem the precompactness of $\bar{Y} = F(Y_0)$ is followed. The operator F has a fixed point in Y_0 is as consequence of Schauder's fixed point theorem and hence fixed point of F is a mild solution of (2) on $[0, t_1]$ satisfying $u(t)\epsilon U$ for $0 \leq t \leq t_1$. This completes the proof of Theorem 2.4.

It is to be noted that a mild solution u of (2) defined on a closed interval $[0, t_1]$ can be extended to a larger interval $[0, t_1 + \delta]$, for some $\delta > 0$, by defining

$$u(t + t_1) = w(t),$$

where $w(t)$ is a mild solution of

$$w''(t) - (A_1 + A_2)w(t) + A_1A_2w(t) = \int_0^{t+t_1} [a(t+t_1, s)g_0(s, w(s))$$
$$+ g_1(t+t_1, s, w(s))]ds + f(t+t_1, w(t)),$$
$$w(0) = u(t_1) \text{ and } w'(0) = u'(t_1). \quad (15)$$

The assurance of the existence of the mild solution of (15) on an interval of positive length $\delta > 0$ is due to Theorem 2.4. Let $[o, t_{max})$ be the maximal interval to which the mild solution u of (2) can be extended; we shall show that if $t_{max} < \infty$ then $||u(t)|| \to \infty$ as $t \uparrow t_{max}$. First we will prove that $t_{max} < \infty$ implies

$\lim_{t\uparrow t_{max}}||u(t)|| = \infty$. Indeed if $t_{max} < \infty$ and $\lim_{t\uparrow t_{max}}||u(t)|| < \infty$, we can assume $||Ti(t)|| \leq M_i, i = 1, 2$ and $||u(t)|| \leq K$, for $0 < t < t_{max}$ where $M_i; i = 1, 2$, and K are constants. By our assumptions on the functions g_0, f and g_1 we also have constants N_1, N_2 and N_3 such that

$$||g_0(s, u(s))|| \leq N_1; ||g_1(t, s, u(s))|| \leq N_2 \; and \; ||f(s, u(s))|| \leq N_3;$$

for $0 \leq s \leq t < t_{max}$. Now, if $0 < \sigma < t < t' < t_{max}$, we have

$$\begin{aligned} u(t') = T_1(t')\phi + \int_0^{t'} T_1(t'-\tau)T_2(\tau)(\psi - A_1\phi)d\tau \\ + \int_0^{t'}\int_0^{\tau} T_1(t'-\tau)T_2(\tau - s)\int_0^s [a(s,\eta)g_0(\eta, u(\eta)) + g_1(s,\eta,u(\eta))]d\eta ds d\tau \\ + \int_0^{t'}\int_0^{\tau} T_1(t'-\tau)T_2(\tau-s)f(s,u(s))dsd\tau \end{aligned}$$

$$\begin{aligned} ||u(t')-u(t)|| \leq ||(T_1(t')-T_1(t))\phi|| + \int_0^t ||T_1(t'-\tau)-T_1(t-\tau)||||T_2(\tau)||||\psi - A_1\phi||d\tau \\ + \int_0^t\int_0^{\tau} ||T_1(t'-\tau)-T_1(t-\tau)||||T_2(\tau-s)||\int_0^s ||a(s,\eta)g_0(\eta,u(\eta)) + g_1(s,\eta,u(\eta))d\eta ds d\tau \\ + \int_0^t\int_0^{\tau} ||T_1(t'-\tau)-T_1(t-\tau)||||T_2(\tau-s)||||f(s,u(s))||dsd\tau \\ + \int_t^{t'} ||T_1(t'-\tau)||||T_2(\tau)||||\psi - A_1\phi||d\tau \\ + \int_t^{t'}\int_0^{\tau} ||T_1(t'-\tau)||||T_2(\tau-s)||\int_0^s ||a(s,\eta)g_0(\eta,u(\eta)) + g_1(s,\eta,u(\eta))||d\eta ds d\tau \\ + \int_t^{t'}\int_0^{\tau} ||T_1(t'-\tau)||||T_2(\tau-s)||||f(s,u(s))||dsd\tau \end{aligned} \tag{16}$$

$$\begin{aligned} \leq ||T_1(t')\phi - T_1(t)\phi|| + \int_0^t M_2||T_1(t'-\tau) - T_1(t-\tau)||||\psi - A_1\phi||d\tau \\ + \int_0^t\int_0^{\tau} M_2||T_1(t'-\tau) - T_1(t-\tau)||\int_0^s (MN_1 + N_2)d\eta ds d\tau \\ + \int_0^t\int_0^{\tau} M_2N_3||T_1(t'-\tau) - T_1(t-\tau)||dsd\tau + M_1M_2(t'-t)[||\psi - A_1\phi|| \\ + (MN_1 + N_2)(t'^2 + t't + t^2) + N_3(t' + t)] \end{aligned}$$

$$\leq ||T_1(t')\phi - T_1(t)\phi|| + [(\int_0^{t-\sigma} + \int_{t-\sigma}^{t})M_2||T_1(t'-\tau) - T_1(t-\tau)||||\psi - A_1\phi||d\tau]$$
$$+ [(\int_0^{t-\sigma} + \int_{t-\sigma}^{t})\int_0^{\tau}\int_0^{s} M_2||T_1(t'-\tau) - T_1(t-\tau)||(MN_1 + N_2)d\eta ds d\tau]$$
$$+ [(\int_0^{t-\sigma} + \int_{t-\sigma}^{t})\int_0^{\tau} M_3N_3||T_1(t'-\tau) - T_1(t-\tau)||ds d\tau] + M_1M_2(t'-t)[||\psi - A_1\phi||$$
$$+ (MN_1 + N_2)(t'^2 + t't + t^2) + N_3(t'+t)]$$

$$= ||T_1(t')\phi - T_1(t)\phi|| + \int_0^{t-\sigma} M_2||T_1(t'-\tau) - T_1(t-\tau)||||\psi - A_1\phi||d\tau$$
$$+ \int_0^{t-\sigma}\int_0^{\tau} M_2||T_1(t'-\tau) - T_1(t-\tau)||\int_0^{s}(MN_1 + N_2)d\eta ds d\tau$$
$$+ \int_0^{t-\sigma}\int_0^{\tau} M_2||T_1(t'-\tau) - T_1(t-\tau)||N_3 ds d\tau$$
$$+ \int_{t-\sigma}^{t} M_2||T_1(t'-\tau) - T_1(t-\tau)||||\psi - A_1\phi||d\tau$$
$$+ \int_{t-\sigma}^{t}\int_0^{\tau} M_2||T_1(t'-\tau) - T_1(t-\tau)||\int_0^{s}(MN_1 + N_2)d\eta ds d\tau$$
$$+ \int_{t-\sigma}^{t}\int_0^{\tau} N_3M_2||T_1(t'-\tau) - T_1(t-\tau)||ds d\tau$$
$$+ M_1M_2(t'-t)[||\psi - A_1\phi|| + (MN_1 + N_2)(t'^2 + t't + t^2) + N_3(t'+t)]$$

Since $T_i(t), i = 1, 2,$ is continuous in the uniform topology for arbitrary $t \geq \sigma > 0$, the right hand side of (16) tends to zero as t and t' tend to t_{max}. Therefore $\lim_{t\uparrow t_{max}} u(t) = u(t_{max})$ exists and by the first part of the proof the solution u can be extended beyond t_{max} which contradicts the maximality of t_{max} which contradicts and so assumption that $t_{max} < \infty$, implies that $\lim_{t\uparrow t_{max}} ||u(t)|| = \infty$, if this is false then there is a sequence $t_n \uparrow t_{max}$ and a constant K such that $||u(t_n)|| \leq K$ for all n.

Let $||T_i(t)|| \leq M; i = 1, 2,$ for $0 \leq t \leq t_{max}$ and let,

$$N_1 = \sup\{||g_0(t, x)|| : 0 \leq t \leq t_{max}||x|| \leq M(K+1)\},$$

$$N_2 = \sup\{||g_1(t, s, x)|| : 0 \leq s, t \leq t_{max}||x|| \leq M(K+1)\}$$

and

$$N_3 = \sup\{||f(t, x)|| : 0 \leq t \leq t_{max}||x|| \leq M(K+1)\}.$$

Since $t \to ||u(t)||$ is continuous and $\lim_{t\uparrow t_{max}} ||u(t)|| = \infty$. We can find a sequence $\{h_n\}$ with the following properties:

$h_n \to 0$ as $n \to \infty$,

$$||u(t)|| \leq M(K+1) \ for \ t_n \leq t \leq t_n + h_n \ and \ ||u(t_n + h_n)|| = M(K+1)$$

.

But then we have

$$M(K+1) = ||u(t_n + h_n)||$$

$$\leq ||T_1(h_n)u(t_n)|| + \int_{t_n}^{t_n+h_n} ||T_1(t_n + h_n - \tau)T_2(\tau)(\psi - A_1\phi)||d\tau$$

$$+ \int_{t_n}^{t_n+h_n} \int_{t_n}^{\tau} \int_{t_n}^{s} ||T_1(t_n + h_n - \tau)T_2(\tau - s)[a(s,\eta)g_0(\eta, u(\eta)) + g_1(s, \eta, u(\eta))]||dsd\tau$$

$$+ \int_{t_n}^{t_n+h_n} \int_{t_n}^{\tau} ||T_1(t_n + h_n - \tau)T_2(\tau - s)f(s, u(s))||dsd\tau$$

$$\leq MK + M^2||\psi - A_1\phi||h_n + M^2(\beta N_1 + N_2)\frac{h_n^3}{3!} + M^2 N_3 \frac{h_n^2}{2!}$$

$$= MK + M^2 h_n[||\psi - A_1\phi|| + (\beta N_1 + N_2)\frac{h_n^2}{3!} + N_3 h/2!].$$

Where $\beta = \max |a(s, \eta)|$, which is absurd as $h_n \to 0$. Therefore, we have

$$\lim_{t \uparrow t_{max}} ||u(t)|| = \infty$$

and the proof of the theorem is complete. □

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