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On closedness of convolution of two sets

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Abstract. In this note, we give an abstract version of the fact that convolution of two closed and compact subsets of a hypergroup is a closed set.

1. Introduction and Preliminaries

It is well-known that if B and C are closed and compact subsets of a locally compact group $(G, \cdot)$, respectively, then the product $B \cdot C := \{x \cdot y : x \in B, y \in C\}$ is closed in G, although the product of two closed subsets of G is not closed in general. Let X and Y be two non-empty sets and $A \subseteq X \times Y$. We denote

$$\pi_X(A) := \{x \in X : \text{for some } y \in Y, (x, y) \in A\}$$

and

$$\pi_Y(A) := \{y \in Y : \text{for some } x \in X, (x, y) \in A\}.$$

H. Przybycień in [5] gave the following abstract version of the above fact.

Theorem 1.1. *Let X, Y and Z be Hausdorff topological spaces and $f : X \times Y \to Z$ be a continuous function such that:*

(1) *for every $y \in Y$ the function $f(\cdot, y)$ is an injection,*
(2) *there exists a continuous function $\varphi : Y \times Z \to X$ such that $f(\varphi(y, z), y) = z$ for all $(y, z) \in Y \times Z$.*

If $A \subseteq X \times Y$ is a closed set such that $\pi_Y(A)$ is compact, then the image $f(A)$ is closed in Z.

Our motivation for writing this work is to give an improvement of this theorem. For this, we need to recall a special topology on the family of all non-empty compact subsets of a topological space which was defined in [6] and studied in [4] too.

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Definition 1.1. Let (X, τ) be a Hausdorff topological space. The family of all non-empty compact subsets of X is denoted by $\mathcal{K}(X)$. For each $A, B \subseteq X$ we denote

$$V_A(B) := \{D \in \mathcal{K}(X) : \ D \subseteq B, \ D \cap A \neq \varnothing\}.$$

Then, the topology generated by the subbasis

$$\{V_A(B) : \ A, B \subseteq X \text{ are open}\}$$

is called *Michael topology* in [3] and is denoted by 2^τ.

Michael topology plays a key role in theory of harmonic analysis on locally compact hypergroups.

Let K be a locally compact Hausdorff space. We denote the space of all bounded Radon measures on K by $\mathcal{M}(K)$, and the set of non-negative elements of $\mathcal{M}(K)$ is denoted by $\mathcal{M}^+(K)$. The support of each measure $\mu \in \mathcal{M}(K)$ is denoted by $\text{supp}\mu$. Also, the Dirac measure at the point $x \in K$ is denoted by δ_x.

Definition 1.2. Let K be a locally compact Hausdorff space with the following property:

(1) there is a mapping $* : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathcal{M}(K)$ (called *convolution*) such that $(\mathcal{M}(K), *, +)$ is a complex Banach algebra;
(2) for each $\mu, \nu \in \mathcal{M}^+(K)$, $\mu * \nu$ is a non-negative measure in $\mathcal{M}(K)$ and the mapping $(\mu, \nu) \mapsto \mu * \nu$ from $\mathcal{M}^+(K) \times \mathcal{M}^+(K)$ into $\mathcal{M}^+(K)$ is continuous, where $\mathcal{M}^+(K)$ is equipped with the cone topology;
(3) for all $x, y \in K$, $\delta_x * \delta_y$ is a compact supported probability measure;
(4) the mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathcal{K}(K)$ equipped with the Michael topology, is continuous;
(5) there is an element e such that for each $x \in K$, $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$;
(6) there is a homeomorphism $x \mapsto x^-$ from K onto K (called *involution*) such that for each $x, y \in K$ we have $(x^-)^- = x$ and $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$;
(7) for each $x, y \in K$, $e \in \text{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$.

Then, $K \equiv (K, *, .^-, e)$ is called a *(locally compact) hypergroup.*

For each subsets A, B of a locally compact hypergroup K we denote

$$A * B := \bigcup_{x \in A, \, y \in B} \text{supp}(\delta_x * \delta_y).$$

2. Main Result

By identifying x and the singleton $\{x\}$, one can easily see that the theorem below is a generalization of Theorem 1.1. This theorem gives also some sufficient conditions for union of a collection of compact sets to be closed. Here, the family of all non-empty compact subsets of a Hausdorff topological space is equipped with Michael topology.

Theorem 2.1. *Let X, Y and Z be locally compact Hausdorff spaces and $f : X \times Y \to \mathcal{K}(Z)$ be a function. Let there exists a continuous function $\phi : Y \times Z \to \mathcal{K}(X)$ such that for each $x \in X$, $y \in Y$ and $z \in Z$,*

$$z \in f(x, y) \quad \text{if and only if} \quad x \in \phi(y, z). \tag{1}$$

Let $A \subseteq X \times Y$ be a rectangle such that $\pi_X(A)$ is a closed subset of X and $\pi_Y(A)$ is a compact subset of Y. Then, $\bigcup_{(x,y)\in A} f(x, y)$ is a closed subset of Z. (We have considered the product topology on $X \times Y$ and $Y \times Z$.)

Proof. Suppose that (z_α) is a net in $\bigcup_{(x,y)\in A} f(x, y)$, $z_0 \in Z$ and $z_\alpha \to z_0$ in Z. Then, for each index α there exists (x_α, y_α) in A such that $z_\alpha \in f(x_\alpha, y_\alpha)$, and so (y_α) is a net in $B := \pi_Y(A)$. This also implies that $x_\alpha \in \phi(y_\alpha, z_\alpha)$. Because of compactness of $\pi_Y(A)$, there exist $y_0 \in \pi_Y(A)$ and a subnet (y_β) of (y_α) such that $y_\beta \to y_0$ in Y. Let F be a compact neighborhood of z_0 in Z. Then, there exists some η such that

$$x_\beta \in \phi(y_\beta, z_\beta) \in \phi(B \times F) \tag{2}$$

for all $\beta \geq \eta$. Since ϕ is continuous and $B \times F$ is compact, $\phi(B \times F)$ is compact in $\mathcal{K}(X)$. So, by [3, 2.5F], the set $\bigcup_{y\in B, z\in F} \phi(y, z)$ is a compact subset of X. Therefore, (without loss the generality, by taking a subnet) there exists an element $x_0 \in X$ such that $x_\beta \to x_0$. Since x_β belongs to the closed set $\pi_X(A)$ for all index β, we have $x_0 \in \pi_X(A)$. Since ϕ is continuous, we have

$$\phi(y_\beta, z_\beta) \to \phi(y_0, z_0) \tag{3}$$

in $\mathcal{K}(X)$. We show that $x_0 \in \phi(y_0, z_0)$. In contrast, assume that $x_0 \notin \phi(y_0, z_0)$. Since $\phi(y_0, z_0)$ is compact, there is a compact neighborhood E of x_0 in X such that $\phi(y_0, z_0) \cap E = \varnothing$. So, $\phi(y_0, z_0) \in V_X(E^c)$. By (3), there exists some γ such that for each $\beta \geq \gamma$, $\phi(y_\beta, z_\beta) \in V_X(E^c)$. Consequently, $x_\beta \in \phi(y_\beta, z_\beta) \subseteq E^c$. Since E is a neighborhood of x_0, this contradicts the fact $x_\beta \to x_0$. Thus, $x_0 \in \phi(y_0, z_0)$, and equivalently, $z_0 \in f(x_0, y_0)$. This completes the proof. □

The next fact which was given in [3, Lemma 4.1E] would be a direct conclusion of Theorem 2.1.

Corollary 2.2. *Let K be a locally compact hypergroup, C be a compact subset of K, and B be a closed subset of K. Then, $B * C$ is closed in K.*

Proof. In theorem 2.1 put $X = Y = Z := K$, $A := B \times C$, $f(x, y) := \{x\} * \{y\}$ and $\phi(y, z) := \{z\} * \{y^-\}$. Now, by [3, Lemma 4.1B] one can see that the condition (1) in Theorem 2.1 holds, and so the proof is complete. □

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