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An existence result for a class of $(p(x), q(x))$-Laplacian system via sub-supersolution method

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Abstract. This study concerns the existence of positive solution for the following nonlinear boundary value problem

$$
\begin{aligned}
-\Delta_{p(x)} u &= a(x)h(u) + f(v) \quad \text{in } \Omega \\
-\Delta_{q(x)} v &= b(x)k(v) + g(u) \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where $p(x), q(x) \in C^1(\mathbb{R}^N)$ are radial symmetric functions such that $\sup |\nabla p(x)| < \infty$,
$\sup |\nabla q(x)| < \infty$ and $1 < \inf p(x) \le \sup p(x) < \infty, 1 < \inf q(x) \le \sup q(x) < \infty$, and where $-\Delta_{p(x)} u = -\operatorname{div} |\nabla u|^{p(x)-2} \nabla u, -\Delta_{q(x)} v = -\operatorname{div} |\nabla v|^{q(x)-2} \nabla v$ respectively are called $p(x)$-Laplacian and $q(x)$-Laplacian, $\Omega = B(0, R) = \{x||x| < R\}$ is a bounded radial symmetric domain, where $R > 0$ is a sufficiently large constant. We discuss the existence of positive solution via sub-supersolutions without assuming sign conditions on $f(0)$ and $g(0)$.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has been a new and interesting topic. Many results have been obtained on this kind of problems; see for example [**3**, **4**, **5**, **6**, **7**, **8**, **13**]. In [**5**, **6**] Fan and Zhao give the regularity of weak solutions for differential equations

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with nonstandard $p(x)$-growth conditions. Zhang [11] investigated the existence of positive solutions of the system

$$\begin{aligned} -\Delta_{p(x)}u &= f(v) \quad \text{in } \Omega \\ -\Delta_{p(x)}v &= g(u) \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a function, $\Omega \subset \mathbb{R}^N$ is a bounded domain. The operator $-\Delta_{p(x)}u = -\operatorname{div}|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x)$ is a constant p, System (1) is the well-known p-Laplacian system.

There are many papers on the existence of solutions for p-Laplacian elliptic systems, for example [1, 3, 4, 5, 6, 7, 8, 9].

In [9] the authors consider the existence of positive weak solutions for the p-Laplacian problem

$$\begin{aligned} -\Delta_p u &= f(v) \quad \text{in } \Omega \\ -\Delta_p v &= g(u) \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{2}$$

There the first eigenfunctions is used for constructing the subsolution of p-Laplacian problems. Under the condition $\lim_{u\to+\infty} f(M(g(u))^{1/(p-1)}/u^{p-1} = 0$, for all $M > 0$, the authors show the existence of positive solutions for problem (2). In this paper we consider the existence of positive solutions of the system

$$\begin{aligned} -\Delta_{p(x)}u &= F(x,u,v) \quad \text{in } \Omega \\ -\Delta_{q(x)}v &= G(x,u,v) \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3}$$

where $p(x), q(x) \in C^1(\mathbb{R}^N)$ is a function, $F(x,u,v) = [a(x)h(u) + f(v)]$, $G(x,u,v) = [b(x)k(v) + g(u)]$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain. The final conclusion can be done by a standard argument via Perron's method (a method for solving the Dirichlet problem for the Laplace equation based on the properties of subharmonic functions). Perron (see [14]) gave the initial presentation of the method, which was substantially developed by Wiener and Keldysh in [15]).

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [4]). If $\Omega \subset \mathbb{R}^N$ is an open domain, write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\}$$

$h^+ = \sup_{x\in\Omega} h(x)$, $h^- = \inf_{x\in\Omega} h(x)$, for any $h \in C(\Omega)$, $L^{p(x)}(\Omega) = \{u | u$ is a measurable real-valued function, $\int_\Omega |u|^{p(x)}dx < \infty\}$.

Throughout the paper, we will assume that $p, q \in C_+(\Omega)$ and $1 < \inf_{x\in\mathbb{R}^N} p(x) \le \sup_{x\in\mathbb{R}^N} p(x) < N$, $1 < \inf_{x\in\mathbb{R}^N} q(x) \le \sup_{x\in\mathbb{R}^N} q(x) < N$. We introduce the norm on $L^{q(x)}(\Omega)$by

$$|u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive and uniform convex Banach space (see [4, Theorem 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$, and it is equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach space (see [4, Theorem 2.1]). We define

$$(L(u), v) = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall u, v \in W^{1,p(x)}(\Omega),$$

then $L : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ is a continuous, bounded and is a strictly monotone operator, and it is a homeomorphism [7, Theorem 3.11].

if $(u, v) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$, (u, v) is called a weak solution of (**??**); if it satisfies

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx = \int_\Omega \lambda F(x, u, v) \varphi dx, \quad \forall \varphi \in W_0^{1,p(x)}(\Omega),$$
$$\int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx = \int_\Omega \lambda G(x, u, v) \psi dx, \quad \forall \psi \in W_0^{1,q(x)}(\Omega).$$

We make the following assumptions

(H1) $p(x), q(x) \in C^1(\mathbb{R}^N)$ is a radial symmetric and $\sup |\nabla p(x)| < \infty, \sup |\nabla q(x)| < \infty$

(H2) $\Omega = B(0, R) = \{x||x| < R\}$ is a ball, where $R > 0$ is a sufficiently large constant.

(H3) $h, k \in C^1([0, \infty))$ are nonnegative, nondecreasing functions such that

$$\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = 0, \quad \lim_{u\to+\infty} \frac{k(v)}{u^{q^- -1}} = 0\,.$$

(H4) $f, g \in C^1([0, \infty))$ are nondecreasing functions, $\lim_{u\to+\infty} f(u) = +\infty$, $\lim_{u\to+\infty} g(u) = +\infty$, and

$$\lim_{u\to+\infty} \frac{f(M(g(u))^{\frac{1}{q^- -1}})}{u^{p^- -1}} = 0, \quad \forall M > 0\,.$$

(H5) $a, b : [0, +\infty) \to (0, \infty)$ are continuous functions such that $a_1 = \min_{x\in\bar\Omega} a(x)$,$b_1 = \min_{x\in\bar\Omega} b(x)$,$a_2 = \max_{x\in\bar\Omega} a(x)$ and $b_2 = \max_{x\in\bar\Omega} a(x)$.

We shall establish the following result.

2. Main result

Theorem 2.1. *If* (H1)–(H5) *hold, then* (3) *has a positive solution.*

Proof. We establish this theorem by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (3), such that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$. That is (ϕ_1, ϕ_2) and (z_1, z_2) satisfy

$$\begin{aligned}
\int_\Omega |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 \cdot \nabla \varphi dx &\leq \int_\Omega a(x) h(\phi_1) \varphi dx + \int_\Omega f(\phi_2) \varphi dx,\\
\int_\Omega |\nabla \phi_2|^{q(x)-2} \nabla \phi_1 \cdot \nabla \psi dx &\leq \int_\Omega b(x) k(\phi_2) \psi dx + \int_\Omega g(\phi_1) \psi dx,\\
\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi dx &\geq \int_\Omega a(x) h(z_1) \varphi dx + \int_\Omega f(z_2) \varphi dx,\\
\int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi dx &\geq \int_\Omega b(x) k(z_2) \psi dx + \int_\Omega g(z_1) \psi dx,
\end{aligned}$$

for all $(\varphi, \psi) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$ with $\varphi, \psi \geq 0$. Then (3) has a positive solution.

Step 1. We construct a subsolution of (3). Denote

$$\alpha_1 = \frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, \quad R_1 = \frac{R - \alpha_1}{2},$$

$$\alpha_2 = \frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, R_2 = \frac{R - \alpha_2}{2}, b = \min\{h(0)a_1 + f(0), k(0)b_1 + h(0), -1\},$$

and let

$$\phi_1(r) = \begin{cases} e^{-k(r-R)} - 1, & 2R_1 < r \leq R,\\ e^{\alpha k} - 1 + \int_r^{2R_1} (k e^{\alpha k})^{\frac{p(2R_1)-1}{p(r)-1}} & \\ \times [\frac{(2R_1)^{N-1}}{r^{N-1}} \sin(\varepsilon(r - 2R_1) + \frac{\pi}{2})(a_1 + 1)]^{\frac{1}{p(r)-1}} dr, & 2R_1 - \frac{\pi}{2\varepsilon} < r \leq 2R_1,\\ e^{\alpha k} - 1 + \int_{2R_1 - \frac{\pi}{2\varepsilon}}^{2R_1} (k e^{\alpha k})^{\frac{p(2R_1)-1}{p(r)-1}} & \\ \times [\frac{(2R_1)^{N-1}}{r^{N-1}} \sin(\varepsilon_0(r - 2R_1) + \frac{\pi}{2})(a_1 + 1)]^{\frac{1}{p(r)-1}} dr, & r \leq 2R_1 - \frac{\pi}{2\varepsilon},\end{cases}$$

where R_1 is sufficiently large, ε is a small positive constant which satisfies $R_1 \leq 2R_1 - \frac{\pi}{2\varepsilon}$, and let

$$\phi_2(r) = \begin{cases} e^{-k(r-R)} - 1, & 2R_2 < r \leq R, \\ e^{\alpha k} - 1 + \int_r^{2R_2} (ke^{\alpha k})^{\frac{q(2R_2)-1}{q(r)-1}} & \\ \times[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin(\varepsilon(r - 2R_2) + \frac{\pi}{2})(b_1 + 1)]^{\frac{1}{q(r)-1}} dr, & 2R_1 - \frac{\pi}{2\varepsilon} < r \leq 2R_2, \\ e^{\alpha k} - 1 + \int_{2R_2 - \frac{\pi}{2\varepsilon}}^{2R_2} (ke^{\alpha k})^{\frac{q(2R_2)-1}{q(r)-1}} & \\ \times[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin(\varepsilon_0(r - 2R_2) + \frac{\pi}{2})(b_1 + 1)]^{\frac{1}{q(r)-1}} dr, & r \leq 2R_2 - \frac{\pi}{2\varepsilon}, \end{cases}$$

where R_2 is sufficiently large, ε is a small positive constant which satisfies $R_2 \leq 2R_2 - \frac{\pi}{2\varepsilon}$,

In the following, we will prove that (ϕ_1, ϕ_2) is a subsolution of (3). Since

$$\phi_1'(r) = \begin{cases} e^{-k(r-R)} - 1, & 2R_1 < r \leq R, \\ -(ke^{\alpha k})^{\frac{p(2R_1)-1}{p(r)-1}} & \\ \times[\frac{(2R_1)^{N-1}}{r^{N-1}} \sin(\varepsilon_1(r - 2R_1) + \frac{\pi}{2})(a_1 + 1)]^{\frac{1}{p(r)-1}} dr, & 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, \\ 0, & 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}, \end{cases}$$

and

$$\phi_2'(r) = \begin{cases} e^{-k(r-R)} - 1, & 2R_2 < r \leq R, \\ -(ke^{\alpha k})^{\frac{q(2R_1)-1}{q(r)-1}} & \\ \times[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin(\varepsilon_2(r - 2R_2) + \frac{\pi}{2})(b_1 + 1)]^{\frac{1}{q(r)-1}} dr, & 2R_2 - \frac{\pi}{2\varepsilon_2} < r \leq 2R_1, \\ 0, & 0 \leq r \leq 2R_2 - \frac{\pi}{2\varepsilon_2}, \end{cases}$$

it is easy to see that $\phi_1, \phi_2 \geq 0$ is decreasing and $\phi_1, \phi_2 \in C^1([0, R]), \phi_1(x) = \phi_1(|x|) \in C^1(\bar{\Omega})$ and $\phi_2(x) = \phi_2(|x|) \in C^1(\bar{\Omega})$.Let $r = |x|$. By computation,

$$-\Delta_{p(x)}\phi_1 = -\operatorname{div} |\nabla\phi(x)|^{p(x)-2}\nabla\phi(x)) = -(r^{N-1}|\phi'(r)|^{p(r)-2}\phi'(r))'/r^{N-1}.$$

Then

$$-\Delta_{p(x)}\phi_1 = \begin{cases} (ke^{-k(r-R)})^{p(r)-1}\big[-k(p(r) - 1) + p'(r)\ln k & \\ -kp'(r)(r - R) + \frac{N-1}{r}\big], & 2R_1 < r \leq R, \\ \varepsilon_1(\frac{2R_1}{r})^{N-1}(ke^{\alpha k})^(p(2R_1) - 1) & \\ \times \cos(\varepsilon_1(r - 2R_1) + \frac{\pi}{2})(a_1 + 1), & 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, \\ 0, & 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}, \end{cases}$$

If k is sufficiently large, when $2R_1 < r \leq R$, then

$$-\Delta_{p(x)}\phi_1 \leq -k[\inf p(x) - 1 - \sup |\nabla p(x)|(\frac{\ln k}{k} + R - r) + \frac{N-1}{kr}] \leq -k\alpha_1.$$

Since α_1 is a constant dependent only on $p(x)$, if k is a big enough, such that $-k\alpha_1 < b$, and since $\phi(x) \geq 0$ and h, f are monotone, this implies

$$-\Delta_{p(x)}\phi_1 \leq h(0)a_1 + f(0) \leq a(x)h(\phi_1) + f(\phi_1), \quad 2R_1 < |x| \leq R\,. \tag{4}$$

If k is sufficiently large, then

$$h(e^{a_2k} - 1) \geq 1, \quad f(e^{a_1k} - 1) \geq 1, \quad k(e^{a_1k} - 1) \geq 1, \quad g(e^{a_2k} - 1) \geq 1$$

where k is dependent on h, f, k, g and p, q and independent on R. Since

$$\begin{aligned}-\Delta_{p(x)}\phi_1 &= \varepsilon_1(\frac{2R_1}{r})^{N-1}(ke^{a_1k})^(p(2R_1) - 1)\cos(\varepsilon_1(r - 2R_1) + \frac{\pi}{2})(a_1 + 1)\\ &\leq \varepsilon_1(a_1 + 1)2^N k^{p^+} e^{a_1kp^+}, 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| < 2R_1\,.\end{aligned}$$

Let $\varepsilon_1 = 2^{-N}k^{-p^+}e^{-a_1kp^+}$. Then

$$-\Delta_{p(x)}\phi \leq a_1 + 1 \leq a(x)h(\phi_1) + f(\phi_1), 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| < 2R_1. \tag{5}$$

Obviously,

$$-\Delta_{p(x)}\phi_1 = 0 \leq a_1 + 1 \leq a(x)h(\phi) + f(\phi), |x| < 2R_1 - \frac{\pi}{2\varepsilon_1}. \tag{6}$$

Since $\phi(x) \in C^1(\Omega)$, combining (4), (5), (6), we have

$$-\Delta_{p(x)}\phi_1 \leq a(x)h(\phi) + f(\phi)$$

for a.e. $x \in \Omega$. Similarly we have

$$-\Delta_{q(x)}\phi_2 \leq b(x)k(\phi) + g(\phi_2),$$

for a.e. $x \in \Omega$. since $\phi_1(x), \phi_2(x) \in C^1(\bar{\Omega})$, it is easy to see that (ϕ_1, ϕ_2) is a subsolution of (3).

Step 2. We construct a supersolution of (3) Let z_1 be a radial solution of

$$\begin{gathered}-\Delta_{p(x)}z_1(x) = (a_2 + 1)\mu, \quad \text{in } \Omega,\\ z_1 = 0 \quad \text{on } \partial\Omega\,.\end{gathered}$$

We denote $z_1 = z_1(r) = z_1(|x|)$, then z_1 satisfies

$$-(r^{N-1}|z_1'|^{p(r)-2}z_1')' = r^{N-1}(a_2 + 1)\mu, z_1(R) = 0, z_1'(0) = 0\,.$$

Then

$$z_1' = -|\frac{r(a_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}}, \tag{7}$$

and

$$z_1 = \int_r^R |\frac{r(a_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}} dr.$$

We denote $\beta = \beta((a_2 + 1)\mu) = \max_{0 \leq r \leq R} z_1(r)$, then

$$\beta((a_2 + 1)\mu) = \int_0^R |\frac{r(a_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}} dr = ((a_2 + 1)\mu)^{\frac{1}{p(q)-1}} \int_0^R |\frac{r}{N}|^{\frac{1}{p(r)-1}} dr,$$

where $q \in [0,1]$. Since $\int_0^R |\frac{r}{N}|^{\frac{1}{p(r)-1}} dr$ is a constant, then there exists a positive constant $C \geq 1$ such that

$$\frac{1}{C}((a_2+1)\mu)^{\frac{1}{p^+-1}} \leq \beta((a_2+1)\mu) = \max_{0\leq r\leq R} z_1(r) \leq C((a_2+1)\mu)^{\frac{1}{p^--1}}. \tag{8}$$

We consider

$$\begin{gathered} -\Delta_{p(x)} z_1 = (a_2+1)\mu \quad \text{in } \Omega \\ -\Delta_{q(x)} z_2 = (b_2+1)h(\beta((a_2+1)\mu)) \quad \text{in } \Omega \\ z_1 = z_2 = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Then we shall prove that (z_1,z_2) is a supersolution for (3). For $\psi \in W^{1,q(x)}(\Omega)$ with $\psi \geq 0$, it is easy to see that

$$\begin{aligned} \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla \psi dx &= \int_\Omega (b_2+1)h(\beta((a_2+1)\mu))\psi dx \\ &\geq \int_\Omega b_2 h(\beta((a_2+1)\mu)\psi dx + \int_\Omega h(z_1)\psi dx. \end{aligned}$$

Similar to (8), we have

$$\max_{0\leq r\leq R} z_2(r) \leq C[(b_2+1)h(\beta((a_2+1)\mu))]^{\frac{1}{(q^--1)}}.$$

By (H3), for μ large enough we have

$$h(\beta((a_2+1)\mu)) \geq b(C[(b_2+1)h(\beta((a_2+1)\mu))]^{\frac{1}{q^--1}}) \geq b(z_2).$$

Hence

$$\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla \psi dx \geq \int_\Omega b(x)k(z_2)\psi dx + \int_\Omega h(z_1)\psi dx, \tag{9}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \varphi dx = \int_\Omega (a_2+1)\mu\varphi dx.$$

By (H3), (H4), when μ is sufficiently large, according to (8), we have

$$\begin{aligned} (a_2+1)\mu &\geq [\frac{1}{C}\beta((a_2+1)\mu)]^{p^--1} \\ &\geq b_2 a(\beta((a_2+1)\mu)) + f[C[(a_2+1)^{\frac{1}{(p^--1)}}(h(\beta((a_2+1)\mu)))^{\frac{1}{(p^--1)}}] \\ &\geq b(x)a(z_1) + f(z_2), \end{aligned}$$

then

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \varphi dx \geq \int_\Omega a(x)h(z_1)\varphi dx + \int_\Omega f(z_2)\varphi dx. \tag{10}$$

According to (9) and (10), we can conclude that (z_1,z_2) is a supersolution of (3).

Let μ be sufficiently large, then from (7) and the definition of (ϕ_1,ϕ_2), it is easy to see that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$. This completes the proof. $\square$

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