Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 3, 2021, 65–82
doi: 10.30495/maca.2021.1937063.1023
ISSN 2716-9898

Controlled g-frames in Hilbert C*-modules

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Abstract. The controlled frame was introduced by Balazs et al. [2], with the aim to improve the efficiency of the iterative algorithms constructed for inverting the frame operator. In this paper, the concept of controlled g-frames is introduced in Hilbert C*-modules. The equivalent condition for controlled g-frame is established using the operator theoretic approach. Some characterizations of controlled g-frames and controlled g-Bessel sequences are found out. Moreover, the relationship between g-frames and controlled g-frames are established. At the end, some perturbation results on controlled g-frames are proved.

1. Introduction

The notion of frames was initiated by Duffin and Schaeffer [6] in 1952, while studying the nonharmonic Fourier series. After a long gap, in 1986, Daubechies et al. [5] reintroduced the same notion and developed the theory of frames. In general, frame is nothing but a spanning set and what makes it interesting is the redundancy. Due to its redundancy it becomes more applicable not only in theoretical point of view but also in various kinds of applications. Due to their rich structure the subject draws the attention of many mathematicians, physicists and engineers since it is largely applicable in signal processing [9], image processing [4], coding and communications [23], sampling [7, 8], numerical analysis, filter theory [3]. Now a days it is used in compressive sensing, data analysis and other areas.

Hilbert C^*-module is a wide category between Hilbert space and Banach space. It was Frank and Larson [11], who initiated the theory of frames in Hilbert C^*-modules. Prior to this, Frank and Larson in [10], generalized the notion of orthonormal bases in Hilbert C*-modules. For more details of frames in Hilbert C^*-modules one may refer to the Doctoral Dissertation [16], Han et al. [13], Han et al. [14] and the references there in. The notion of g-frame or generalized frame in Hilbert C^*-module is introduced by Sun [25]. For more on g-frames one can refer to Khosravi and

2010 *Mathematics Subject Classification.* 42C15; 06D22; 46H35.
Key words and phrases. Frame, g-frame, Hilbert C*-module

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Khosravi [18], Fu and Zhang [12], Li and Leng [22]. To improve the numerical efficiency of iterative algorithms for inverting the frame operator, controlled frame was introduced by Balazs et al. [2] in Hilbert spaces. Recently, Kouchi and Rahimi [19] introduced Controlled frames in Hilbert C^*-modules. Controlled fusion frames have been investigated in [20]. The notion of *-Controlled frames can be found in Shateri [24]. Kabbaj et al. [17] introduced controlled continuous g-frames in Hilbert C*-modules. Motivated from the above literature, we study the notion of controlled g-frame in Hilbert C^*-modules. The investigation is quite different from the work of Kabbaj et al. [17].

2. Preliminaries

Let us briefly recall some definitions and basic properties of Hilbert C^* modules. Hilbert C^*-modules are generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than the usual fields $\mathbb{R}$ or $\mathbb{C}$.

Let $\mathcal{A}$ be a unital C^*-algebra. The Hilbert C^*-module or Hilbert $\mathcal{A}$-module is defined as follows:

Definition 2.1. Let $\mathcal{H}$ be a left $\mathcal{A}$-module. $\mathcal{H}$ is called a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that
(i) $\langle f, f \rangle \geq 0$,$\forall f \in \mathcal{H}$, and $\langle f, f \rangle = 0$ if and only if $f = 0$.
(ii) $\langle f, g \rangle = \langle g, f \rangle^*$.
(iii) $\langle af + g, h \rangle = a\langle f, h \rangle + \langle g, h \rangle$ for all $f, g, h \in \mathcal{H}$ and $a \in \mathcal{A}$.

For every $f \in \mathcal{H}$, the norm is defined as $\|f\| = \|\langle f, f \rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with respect to the norm, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. Frame in Hilbert C^*-module is defined as:

Definition 2.2. [11] A family of elements $\{f_j\}_{j\in\mathbb{J}}$ in a Hilbert C^*-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$, is said to be a frame if there exist two constants $C, D > 0$ such that

$$C\langle f, f \rangle \leq \sum_{j\in\mathbb{J}} \langle f, f_j \rangle \langle f_j, f \rangle \leq D\langle f, f \rangle, \forall f \in \mathcal{H}.$$

After the introduction of frames in Hilbert C^*-modules, a lot of work on frame theory has been done in Hilbert C^*-modules. The concept of g-frames by Sun [25] in Hilbert C^*-modules is defined as follows:

Definition 2.3. Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$, $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ be a sequence of closed submodules of $\mathcal{H}$. A sequence $\{\wedge_j \in End^*_A(\mathcal{H}, \mathcal{H}_j)\}$ is called a g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ if there exist positive constants C, D such that

$$C\langle f, f \rangle \leq \sum_{j\in\mathbb{J}} \langle \wedge_j f, \wedge_j f \rangle \leq D\langle f, f \rangle, \forall f \in \mathcal{H}. \tag{1}$$

The g-frame operator $S_\Lambda : H \to H$ is defined as

$$S_\Lambda f = \sum_{j\in J} \Lambda_j^* \Lambda_j f. \tag{2}$$

Very recently, controlled frames in Hilbert C^*-modules is introduced by Kouchi and Rahimi [19]. It is defined as follow:

Definition 2.4. Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$. A family of vectors $\{f_j \in \mathcal{H}\}_{j\in\mathbb{J}}$ is said to be a controlled frame in $\mathcal{H}$ or C-Controlled frame in $\mathcal{H}$ if there exist constants $m, M > 0$ such that

$$m\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle f, f_j\rangle\langle Cf_j, f\rangle \leq M\langle f, f\rangle, \forall f \in \mathcal{H}.$$

In the next section we introduce the notion of Controlled g-frames in Hilbert C^*-modules. We study several characterizations of a controlled g-frame, equivalent formulation, its operator theoretic behavior, its relationship with the frames etc.. At the end, we present some stability results on controlled g-frames.

3. Controlled g-frame

Let $\mathcal{H}$ be a C^*-module over a unital C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-valued inner product $\langle ., .\rangle$ and norm $\|.\|$. Let $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse. Let $\{\mathcal{H}_j\}_{j\in J}$ be a sequence of closed submodules of $\mathcal{H}$, where J is any index set. For each $j \in J$, $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{H}_j$.
Let us denote

$$\bigoplus_{j\in J} \mathcal{H}_j = \Big\{ g = \{g_j\}_{j\in J} : \ g_j \in \mathcal{H}_j \text{ and } \sum_{j\in J} \langle g_j, g_j\rangle \text{ is norm convergent in } \mathcal{A} \Big\}.$$

For $f = \{f_j\}_{j\in J},\ g = \{g_j\}_{j\in J}$, the $\mathcal{A}$-valued inner product in $\bigoplus_{j\in J} \mathcal{H}_j$ is defined as

$$\langle f, g\rangle = \sum_{j\in J} \langle f_j, g_j\rangle.$$

The norm is defined by $\|g\| = \|\langle g, g\rangle\|^{\frac{1}{2}} = \big\| \sum_{j\in J} \langle g_j, g_j\rangle \big\|^{\frac{1}{2}}$. With this norm and inner product, $\bigoplus_{j\in J} \mathcal{H}_j$ is also a Hilbert C^*-module over the C^* algebra $\mathcal{A}$.

For the above literature one may refer Lance [21] and Xiao and Zeng [26].

We define below the (C, C')-controlled g-frame.

Definition 3.1. Let $C, C' \in GL^+(\mathcal{H})$. A sequence $\{\Lambda_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ is said to be a (C, C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\langle f, f\rangle \leq \sum_{j\in J} \langle \Lambda_j Cf, \Lambda_j C' f\rangle \leq B\langle f, f\rangle,\ \forall f \in \mathcal{H}. \tag{3}$$

When $A = B$, the sequence $\{\Lambda_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ is called (C, C')-controlled tight g-frame, and when $A = B = 1$, it is called a (C, C')-controlled Parseval g-frame.

Definition 3.2. A sequence $\{\Lambda_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ is said to be a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if there exists constant $0 < B < \infty$ such that

$$\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \leq B\langle f, f\rangle, \ \forall f \in \mathcal{H}. \tag{4}$$

Example 3.3. Let $\mathcal{H}$ be an ordinary inner product space, $J = \mathbb{N}$, and $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for Hilbert $\mathcal{C}$-module $\mathcal{H}$. We construct $\mathcal{H}_j$ as $\mathcal{H}_j = \overline{span}\{e_1, e_2, ..., e_j\}$ for each $j \in \mathbb{N}$.
Define $\Lambda_j : \mathcal{H} \to \mathcal{H}_j$ by

$$\Lambda_j f = \sum_{k=1}^{j} \langle f, \frac{e_j}{\sqrt{j}}\rangle e_k.$$

The adjoint operator $\Lambda_j^* : \mathcal{H}_j \to \mathcal{H}$ can be easily found as

$$\Lambda_j^*(g) = \sum_{k=1}^{j} \langle f, \frac{e_k}{\sqrt{j}}\rangle e_j.$$

Let us define $Cf = 2f$ and $C'f = \frac{1}{2}f$. Then for any $f \in \mathcal{H}$, we can estimate

$$\begin{aligned}
\Big\langle \Lambda_j Cf, \Lambda_j C'f \Big\rangle &= \Big\langle \sum_{k=1}^{j} \Big\langle 2f, \frac{e_j}{\sqrt{j}}\Big\rangle e_k, \sum_{k=1}^{j} \Big\langle \frac{1}{2}f, \frac{e_j}{\sqrt{j}}\Big\rangle e_k \Big\rangle \\
&= \Big\langle 2f, \frac{e_j}{\sqrt{j}}\Big\rangle \Big\langle \frac{1}{2}f, \frac{e_j}{\sqrt{j}}\Big\rangle^* \sum_{k=1}^{j} \|e_k\|^2 \\
&= \Big\langle 2f, \frac{e_j}{\sqrt{j}}\Big\rangle \Big\langle \frac{1}{2}f, \frac{e_j}{\sqrt{j}}\Big\rangle^* j \\
&= \langle f, e_j\rangle \langle f, e_j\rangle^*.
\end{aligned}$$

Therefore, for any $f \in \mathcal{H}$,

$$\sum_{j=1}^{\infty} \Big\langle \Lambda_j Cf, \Lambda_j C'f \Big\rangle = \sum_{j=1}^{\infty} \langle f, e_j\rangle \langle f, e_j\rangle^* = \langle f, f\rangle.$$

This shows that $\{\Lambda_j : j \in \mathbb{N}\}$ is a (C, C')-controlled Parseval g-frame for $\mathcal{H}$ with respect to $\{H_j\}_{j\in\mathbb{N}}$.

Suppose that $\{\Lambda_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ be a (C, C')-controlled g-frame for the Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. The bounded linear operator

$T_{(C,C')}:\bigoplus_{j\in J}\mathcal{H}_j\to\mathcal{H}$ defined by

$$T_{(C,C')}(\{g_j\}_{j\in J})=\sum_{j\in J}(CC')^{\frac{1}{2}}\Lambda_j^*g_j,\ \forall\{g_j\}_{j\in J}\in\bigoplus_{j\in J}\mathcal{H}_j \tag{5}$$

is called the synthesis operator for the (C,C')-controlled g-frame $\{\Lambda_j:j\in J\}$. When C and C' commute with each other, and commute with the operator $\Lambda_j^*\Lambda_j$ for each j, the adjoint operator $T^*_{(C,C')}:\mathcal{H}\to\bigoplus_{j\in J}\mathcal{H}_j$ given by

$$T^*_{(C,C')}(f)=\left\{\Lambda_j(C'C)^{\frac{1}{2}}f\right\}_{j\in J} \tag{6}$$

is called the analysis operator for the (C,C')-controlled g-frame $\{\Lambda_j:j\in J\}$. The (C,C')-controlled g-frame operator $S_{(C,C')}:\mathcal{H}\to\mathcal{H}$ is defined as

$$S_{(C,C')}f = T_{(C,C')}T^*_{(C,C')}f=\sum_{j\in J}C'\Lambda_j^*\Lambda_jCf. \tag{7}$$

For the above discussion one can refer to Hua and Huang [15].

So from now on we assume that C and C' commute with each other, and commute with the operator $\Lambda_j^*\Lambda_j$ for each j.

Proposition 3.1. Let $\{\Lambda_j:j\in J\}$ be a (C,C')-controlled g-frame for the Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Then the (C,C')-controlled g-frame operator $S_{(C,C')}$ is positive, self adjoint and invertible.

Proof. The frame operator $S_{(C,C')}$ for the (C,C')-controlled g-frame is $S_{(C,C')}f=\sum_{j\in J}C'\Lambda_j^*\Lambda_jCf$. As $\{\Lambda_j:j\in J\}$ is a (C,C')-controlled g-frame, and from the following identity,

$$\sum_{j\in J}\langle\Lambda_jCf,\Lambda_jC'f\rangle=\Big\langle\sum_{j\in J}C'\Lambda_j^*\Lambda_jCf,f\Big\rangle=\big\langle S_{(C,C')}f,f\big\rangle,$$

we clearly see that $S_{(C,C')}$ is a positive operator. Also it is clearly bounded and linear. Again

$$\begin{aligned}\big\langle S_{(C,C')}f,g\big\rangle &= \Big\langle\sum_{j\in J}C'\Lambda_j^*\Lambda_jCf,g\Big\rangle\\ &= \sum_{j\in J}\big\langle C'\Lambda_j^*\Lambda_jCf,g\big\rangle\\ &= \sum_{j\in J}\big\langle f,C\Lambda_j^*\Lambda_jC'g\big\rangle=\sum_{j\in J}\big\langle f,S_{(C',C)}g\big\rangle.\end{aligned}$$

Hence $S^*_{(C,C')}=S_{(C',C)}$. Also as C and C' commute with each other and commute with $\Lambda_j^*\Lambda_j$, we have $S_{(C,C')}=S_{(C',C)}$. So the controlled g-frame operator is self adjoint. Alternatively, this can also be directly obtained as $S_{(C,C')}$ is a positive operator, and every positive operator is self adjoint.

From the controlled g-frame identity we have

$$\begin{aligned} & A\langle f,f\rangle \leq \big\langle S_{(C,C')}f,f\big\rangle \leq B\langle f,f\rangle \\ \Rightarrow\quad & A\ Id_{\mathcal{H}} \leq S_{(C,C')} \leq B\ Id_{\mathcal{H}}, \end{aligned}$$

where $Id_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$. Thus the controlled g-frame operator $S_{(C,C')}$ is invertible. □

Lemma 3.2. [1] Let $\mathcal{A}$ be a C^*-algebra. Let U and V be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U,V)$. Then the following statements are equivalent:

(1) T is surjective.
(2) T^* is bounded below with respect to norm i.e there exists $m > 0$ such that $\|T^*f\| \geq m\|f\|$ for all $f \in U$.
(3) T^* is bounded below with respect to inner product i.e there exists $m > 0$ such that $\langle T^*f, T^*f\rangle \geq m\langle f,f\rangle$ for all $f \in U$.

With the help of the above Lemma 3.2, we establish an equivalent definition of (C,C')-controlled g-frame.

Theorem 3.3. Let $\{\Lambda_j : j \in J\} \subset End^*_{A}(\mathcal{H},\mathcal{H}_j)$ and $\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle$ converge in norm for any $f \in \mathcal{H}$. Then $\{\Lambda_j : j \in J\}$ is a $(C,C')-$controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if and only if there exists constants $A, B > 0$ such that

$$A\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\| \leq B\|f\|^2,\ \forall f \in \mathcal{H}. \tag{8}$$

Proof. Let $\{\Lambda_j : j \in J\}$ be a $(C,C')-$controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with bound A and B. Hence we have

$$A\langle f,f\rangle \leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \leq B\langle f,f\rangle,\ \forall f \in \mathcal{H}. \tag{9}$$

Since $\langle f,f\rangle \geq 0,\ \forall f \in \mathcal{H}$, then we can take the norm on the left, middle and right terms of the above inequality (9). Thus we have

$$\begin{aligned} & \|A\langle f,f\rangle\| \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\| \leq \|B\langle f,f\rangle\|,\ \forall f \in \mathcal{H} \\ \Rightarrow\quad & A\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\| \leq B\|f\|^2,\ \forall f \in \mathcal{H}. \end{aligned}$$

Conversely, suppose that

$$A\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\| \leq B\|f\|^2,\ \forall f \in \mathcal{H}. \tag{10}$$

From Proposition (3.1), the $(C,C')-$controlled g-frame operator $S_{(C,C')}$ is positive, self adjoint and invertible. Hence

$$\langle S^{\frac{1}{2}}_{(c,c')}f, S^{\frac{1}{2}}_{(c,c')}f\rangle = \langle S_{(c,c')}f, f\rangle = \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle. \tag{11}$$

Using (11) in (10), we get

$$\sqrt{A}\|f\| \leq \|S^{\frac{1}{2}}_{(c,c')}f\| \leq \sqrt{B}\|f\|, \ \forall f \in \mathcal{H}. \tag{12}$$

According to Lemma 3.2 and inequality (12), there exist constant $m, M > 0$ such that

$$m\langle f, f\rangle \leq \langle S^{\frac{1}{2}}_{(c,c')}f, S^{\frac{1}{2}}_{(c,c')}f\rangle = \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \leq M\langle f, f\rangle, \ \forall f \in \mathcal{H}.$$

Therefore, $\{\Lambda_j : j \in J\}$ is a $(C,C')-$controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. □

Definition 3.4. Let $C \in GL^+(\mathcal{H})$. The sequence $\{\Lambda_j \in End^*_A(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ is said to be a $(C,C)-$controlled g-frame or C^2-controlled g-frame if there exist constants $0 < A \leq B < \infty$ such that

$$A\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle \leq B\langle f, f\rangle, \ \forall f \in \mathcal{H},$$

or equivalently,

$$A\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle\| \leq B\|f\|^2, \ \forall f \in \mathcal{H}. \tag{13}$$

Using some tools from operator algebras, Xiao and Zeng [26] have proved the following equivalent characterization of g-frames in Hilbert C^*-modules.

Theorem 3.4. [26] Let $\{\Lambda_j \in End^*_A(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ and $\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle$ converge in norm for $f \in \mathcal{H}$. Then $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if and only if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \Big\|\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\Big\| \leq B\|f\|^2, \ \forall f \in \mathcal{H}. \tag{14}$$

The above result can be easily seen as a corollary of our Theorem 3.3, when we take $C = C' = I$.

Proposition 3.5. Let $C \in GL^+(\mathcal{H})$. The family $\{\Lambda_j : j \in J\}$ is a g-frame if and only if $\{\Lambda_j : j \in J\}$ is a C^2-controlled g-frame.

Proof. Suppose that $\{\Lambda_j : j \in J\}$ is a C^2-controlled g-frame with bounds A and B. Then from (13), we have

$$A\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle\| \leq B\|f\|^2, \ \forall f \in \mathcal{H}.$$

Now for any $f \in \mathcal{H}$,

$$\begin{aligned} A\|f\|^2 = A\|CC^{-1}f\|^2 &\leq A\|C\|^2\|C^{-1}f\|^2 \\ &\leq \|C\|^2\|\sum_{j\in J}\langle \Lambda_j CC^{-1}f, \Lambda_j CC^{-1}f\rangle\| \\ &= \|C\|^2\|\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\|. \end{aligned}$$

Hence

$$A\|C\|^{-2}\|f\|^2 \leq \|\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\|. \tag{15}$$

Again for any $f \in \mathcal{H}$,

$$\begin{aligned} \|\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\| &= \|\sum_{j\in J}\langle \Lambda_j CC^{-1}f, \Lambda_j CC^{-1}f\rangle\| \\ &\leq B\|C^{-1}f\|^2 \leq B\|C^{-1}\|^2\|f\|^2. \end{aligned} \tag{16}$$

From (15), (16) and Theorem 3.4, we conclude that $\{\Lambda_j : j \in J\}$ is a g-frame with bound $A\|C\|^{-2}$ and $B\|C^{-1}\|^2$.
Conversely, let $\{\Lambda_j : j \in J\}$ is a g-frame with bounds A' and B'. Then for all $f \in \mathcal{H}$,

$$A'\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq B'\langle f, f\rangle.$$

So for $f \in \mathcal{H}$ we have $Cf \in \mathcal{H}$, and

$$\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle \leq B'\langle Cf, Cf\rangle \leq B'\|C\|^2\langle f, f\rangle. \tag{17}$$

Also for any $f \in \mathcal{H}$,

$$\begin{aligned} A'\langle f, f\rangle = A'\langle C^{-1}Cf, C^{-1}Cf\rangle &\leq A'\|C^{-1}\|^2\langle Cf, Cf\rangle \\ &\leq \|C^{-1}\|^2\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle. \end{aligned} \tag{18}$$

From (17) and (18), we have

$$A'\|C^{-1}\|^{-2}\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle \leq B'\|C\|^2\langle f, f\rangle,\ \forall f \in \mathcal{H}.$$

Hence $\{\Lambda_j : j \in J\}$ is a C^2-controlled g-frame with bounds $A'\|C^{-1}\|^{-2}$ and $B'\|C\|^2$. □

Next, we study when a g-Bessel sequence becomes a $(C, C')-$controlled g-Bessel sequence.

Proposition 3.6. Let $\{\Lambda_j : j \in J\}$ is a g-Bessel for the Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Let $C, C' \in GL^+(\mathcal{H})$. Then $\{\Lambda_j : j \in J\}$ is a $(C, C')-$controlled g-Bessel sequence.

Proof. $\{\Lambda_j : j \in J\}$ is a g-Bessel sequence for the Hilbert C^*-module $\mathcal{H}$ with bound B.

$$\begin{aligned}
\|\sum_{j\in J}\langle \Lambda_j C f, \Lambda_j C' f\rangle\| &= \|\sum_{j\in J}\langle C'\Lambda_j^*\Lambda_j C f, f\rangle\| \\
&= \|\langle C'C\sum_{j\in J}\Lambda_j^*\Lambda_j f, f\rangle\| \\
&\leq \|C'C\|\|\langle\sum_{j\in J}\Lambda_j^*\Lambda_j f, f\rangle\| \\
&= \|C'C\|\|\sum_{j\in J}\langle\Lambda_j^*\Lambda_j f, f\rangle\| \\
&= \|C'C\|\|\sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle\| \\
&\leq B\|C'C\|\|f\|^2, \ \forall f \in \mathcal{H}.
\end{aligned}$$

Hence $\{\Lambda_j : j \in J\}$ is a $(C, C')-$controlled g-Bessel sequence with bound $B\|C'C\|$. □

Theorem 3.7. Let $C, C' \in GL^+(\mathcal{H})$, $\{\Lambda_j : j \in J\} \subset End_A^*(\mathcal{H}, \mathcal{H}_j)$, and C, C' commute with each other and commute with $\Lambda_j^*\Lambda_j$ for all $j \in J$. Then the sequence $\{\Lambda_j : j \in J\}$ is a $(C, C')-$controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with bound B if and only if the operator $T_{(C,C')} : \bigoplus_{j\in J}\mathcal{H}_j \to \mathcal{H}$ given by

$$T_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J}(CC')^{\frac{1}{2}}\Lambda_j^* g_j, \ \forall\{g_j\}_{j\in J} \in \bigoplus_{j\in J}\mathcal{H}_j$$

is well defined and bounded operator with $\|T_{(C,C')}\| \leq \sqrt{B}$.

Proof. Let $\{\Lambda_j : j \in J\}$ be a $C, C'-$controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with bound B. As a result of Theorem 3.3,

$$\|\sum_{j\in J}\langle\Lambda_j C f, \Lambda_j C' f\rangle\| \leq B\|f\|^2, \ \forall f \in \mathcal{H}. \tag{19}$$

For any sequence $\{g_j\}_{j\in J} \in \bigoplus_{j\in J} \mathcal{H}_j$,

$$\begin{aligned}
\big\|T_{(C,C')}\big(\{g_j\}_{j\in J}\big)\big\|^2 &= \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\langle T_{(C,C')}\big(\{g_j\}_{j\in J}\big), f\rangle\big\|^2 \\
&= \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\big\langle \sum_{j\in J}(CC')^{\frac{1}{2}}\Lambda_j^* g_j, f\big\rangle\big\|^2 \\
&= \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\sum_{j\in J}\big\langle (CC')^{\frac{1}{2}}\Lambda_j^* g_j, f\big\rangle\big\|^2 \\
&= \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\sum_{j\in J}\big\langle g_j, \Lambda_j(CC')^{\frac{1}{2}} f\big\rangle\big\|^2 \\
&\le \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\sum_{j\in J}\langle g_j, g_j\rangle\big\|\big\|\sum_{j\in J}\big\langle \Lambda_j(CC')^{\frac{1}{2}} f, \Lambda_j(CC')^{\frac{1}{2}} f\big\rangle\big\| \\
&= \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\sum_{j\in J}\langle g_j, g_j\rangle\big\|\big\|\sum_{j\in J}\big\langle \Lambda_j C f, \Lambda_j C' f\big\rangle\big\| \\
&\le \sup_{f\in\mathcal{H},\ \|f\|=1} \big\|\sum_{j\in J}\langle g_j, g_j\rangle\big\| B\|f\|^2 = B\big\|\{g_j\}\big\|^2.
\end{aligned}$$

Therefore, the sum $\sum_{j\in J}(CC')^{\frac{1}{2}}\Lambda_j^* g_j$ is convergent, and we have

$$\begin{aligned}
&\big\|T_{(C,C')}(\{g_j\}_{j\in J})\big\|^2 \le B\big\|\{g_j\}\big\|^2 \\
\Rightarrow\ &\|T_{(C,C')}\| \le \sqrt{B}.
\end{aligned}$$

Hence the operator $T_{(C,C')}$ is well defined, bounded and $\|T_{(C,C')}\| \le \sqrt{B}$.

Conversely, let the operator $T_{(C,C')}$ is well defined, bounded and $\|T_{(C,C')}\| \le \sqrt{B}$. For any $f \in \mathcal{H}$ and finite subset $K \subset J$, we have

$$\begin{aligned}
\big\|\sum_{j\in K}\big\langle \Lambda_j C f, \Lambda_j C' f\big\rangle\big\| &= \big\|\sum_{j\in K}\big\langle C'\Lambda_j^*\Lambda_j C f, f\big\rangle\big\| \\
&= \big\|\sum_{j\in K}\big\langle (CC')^{\frac{1}{2}}\Lambda_j^*\Lambda_j(CC')^{\frac{1}{2}} f, f\big\rangle\big\| \\
&= \Big\|\Big\langle T_{(C,C')}(\{g_j\}_{j\in J}), f\Big\rangle\Big\| \\
&\le \|T_{(C,C')}\|\|\{g_j\}_{j\in J}\|\|f\|,
\end{aligned}$$

where

$$g_j = \begin{cases} \Lambda_j(CC')^{\frac{1}{2}} f, & \text{if } j \in K \\ 0, & j \notin K \end{cases}.$$

Therefore,

$$\begin{aligned}\Big\|\sum_{j\in K}\big\langle \Lambda_j Cf,\Lambda_j C'f\big\rangle\Big\| &\leq \|T_{(C,C')}\|\Big(\Big\|\sum_{j\in K}\big\langle \Lambda_j (CC')^{\frac{1}{2}}f,\Lambda_j (CC')^{\frac{1}{2}}f\big\rangle\Big\|\Big)^{\frac{1}{2}}\|f\| \\ &= \|T_{(C,C')}\|\Big(\Big\|\sum_{j\in K}\big\langle \Lambda_j Cf,\Lambda_j C'f\big\rangle\Big\|\Big)^{\frac{1}{2}}\|f\|.\end{aligned}$$

Since K is arbitrary, we have

$$\begin{aligned}&\Big\|\sum_{j\in J}\big\langle \Lambda_j Cf,\Lambda_j C'f\big\rangle\Big\| \leq \|T_{(C,C')}\|^2\|f\|^2 \\ \Rightarrow\ &\Big\|\sum_{j\in J}\big\langle \Lambda_j Cf,\Lambda_j C'f\big\rangle\Big\| \leq B\|f\|^2,\ \text{as } \|T_{(C,C')}\|\leq\sqrt{B}.\end{aligned}$$

Hence we conclude that $\{\Lambda_j : j\in J\}$ is a $(C,C')-$controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. □

Now we prove some perturbation results for $(C,C')-$controlled g-frame.

Theorem 3.8. Let $\{\Lambda_j\in End_A^*(\mathcal{H},\mathcal{H}_j) : j\in J\}$ be a $(C,C')-$controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Let $\{\Pi_j\in End_A^*(\mathcal{H},\mathcal{H}_j) : j\in J\}$ be any sequence, and assume that C and C' commute with each other and commute with $(\Lambda_j-\Pi_j)^*(\Lambda_j-\Pi_j)$. Then $\{\Pi_j : j\in J\}$ is a $(C,C')-$controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if and only if there exists constants M_1 and M_2 such that

$$\Big\|\sum_{j\in J}\big\langle(\Lambda_j-\Pi_j)Cf,(\Lambda_j-\Pi_j)C'f\big\rangle\Big\|\leq M_1\Big\|\sum_{j\in J}\big\langle\Lambda_j Cf,\Lambda_j C'f\big\rangle\Big\| \tag{20}$$

and

$$\Big\|\sum_{j\in J}\big\langle(\Lambda_j-\Pi_j)Cf,(\Lambda_j-\Pi_j)C'f\big\rangle\Big\|\leq M_2\Big\|\sum_{j\in J}\big\langle\Pi_j Cf,\Pi_j C'f\big\rangle\Big\|. \tag{21}$$

Proof. Let $\{\Lambda_j : j\in J\}$ be a $(C,C')-$controlled g-frame for $\mathcal{H}$ with lower and upper bounds A_1 and B_1, respectively. Also suppose that $\{\Pi_j : j\in J\}$ be a

$(C,C')-$controlled g-frame for $\mathcal{H}$ with lower and upper bounds A_2 and B_2, respectively. Then

$$\begin{aligned}
&\|\sum_{j\in J}\big\langle(\Lambda_j-\Pi_j)Cf,(\Lambda_j-\Pi_j)C'f\big\rangle\| \\
= &\ \|\sum_{j\in J}\big\langle(\Lambda_j-\Pi_j)(CC')^{\frac{1}{2}}f,(\Lambda_j-\Pi_j)(CC')^{\frac{1}{2}}f\big\rangle\| \\
= &\ \big\|\big\{(\Lambda_j-\Pi_j)(CC')^{\frac{1}{2}}f\big\}_{j\in J}\big\|^2 \\
\leq &\ \big\|\big\{\Lambda_j(CC')^{\frac{1}{2}}f\big\}_{j\in J}\big\|^2+\big\|\big\{\Pi_j(CC')^{\frac{1}{2}}f\big\}_{j\in J}\big\|^2 \\
= &\ \|\sum_{j\in J}\big\langle\Lambda_j(CC')^{\frac{1}{2}}f,\Lambda_j(CC')^{\frac{1}{2}}f\big\rangle\|+\|\sum_{j\in J}\big\langle\Pi_j(CC')^{\frac{1}{2}}f,\Pi_j(CC')^{\frac{1}{2}}f\big\rangle\| \\
= &\ \|\sum_{j\in J}\big\langle\Lambda_jCf,\Lambda_jC'f\big\rangle\|+\|\sum_{j\in J}\big\langle\Pi_jCf,\Pi_jC'f\big\rangle\| \\
\leq &\ \|\sum_{j\in J}\big\langle\Lambda_jCf,\Lambda_jC'f\big\rangle\|+B_2\|f\|^2 \\
\leq &\ \|\sum_{j\in J}\big\langle\Lambda_jCf,\Lambda_jC'f\big\rangle\|+\frac{B_2}{A_1}\|\sum_{j\in J}\big\langle\Lambda_jCf,\Lambda_jC'f\big\rangle\| \\
= &\ \Big(1+\frac{B_2}{A_1}\Big)\|\sum_{j\in J}\big\langle\Lambda_jCf,\Lambda_jC'f\big\rangle\|.
\end{aligned}$$

Thus (20) is proved, where $M_1=\Big(1+\frac{B_2}{A_1}\Big)$. In a similar manner, one can obtain

$$\|\sum_{j\in J}\big\langle(\Lambda_j-\Pi_j)Cf,(\Lambda_j-\Pi_j)C'f\big\rangle\|\leq\Big(1+\frac{B_1}{A_2}\Big)\|\sum_{j\in J}\big\langle\Pi_jCf,\Pi_jC'f\big\rangle\|.$$

Hence (21) follows with $M_2=\Big(1+\frac{B_1}{A_2}\Big)$.

Conversely, suppose that $\{\Lambda_j:j\in J\}$ be a $(C,C')-$controlled g-frame for $\mathcal{H}$ with lower and upper bounds A_1 and B_1, respectively, and (20) and (21) hold true.

Then for any $f \in \mathcal{H}$, using (21) we get

$$\begin{aligned}
A_1\|f\|^2 &\leq \Big\|\sum_{j\in J}\big\langle \Lambda_j Cf, \Lambda_j C' f\big\rangle\Big\| \\
&= \Big\|\sum_{j\in J}\big\langle \Lambda_j (CC')^{\frac{1}{2}} f, \Lambda_j (CC')^{\frac{1}{2}} f\big\rangle\Big\| \\
&= \Big\|\big\{\Lambda_j (CC')^{\frac{1}{2}} f\big\}_{j\in J}\Big\|^2 \\
&\leq \Big\|\big\{(\Lambda_j - \Pi_j)(CC')^{\frac{1}{2}} f\big\}_{j\in J}\Big\|^2 + \Big\|\big\{\Pi_j (CC')^{\frac{1}{2}} f\big\}_{j\in J}\Big\|^2 \\
&= \Big\|\sum_{j\in J}\big\langle (\Lambda_j - \Pi_j)(CC')^{\frac{1}{2}} f, (\Lambda_j - \Pi_j)(CC')^{\frac{1}{2}} f\big\rangle\Big\| \\
&\quad + \Big\|\sum_{j\in J}\big\langle \Pi_j (CC')^{\frac{1}{2}} f, \Pi_j (CC')^{\frac{1}{2}} f\big\rangle\Big\| \\
&= \Big\|\sum_{j\in J}\big\langle (\Lambda_j - \Pi_j)Cf, (\Lambda_j - \Pi_j)C' f\big\rangle\Big\| + \Big\|\sum_{j\in J}\big\langle \Pi_j Cf, \Pi_j C' f\big\rangle\Big\| \\
&\leq M_2\Big\|\sum_{j\in J}\big\langle \Pi_j Cf, \Pi_j C' f\big\rangle\Big\| + \Big\|\sum_{j\in J}\big\langle \Pi_j Cf, \Pi_j C' f\big\rangle\Big\| \\
&= (1+M_2)\Big\|\sum_{j\in J}\big\langle \Pi_j Cf, \Pi_j C' f\big\rangle\Big\|.
\end{aligned}$$

This implies that

$$\frac{A_1}{(1+M_2)}\|f\|^2 \leq \Big\|\sum_{j\in J}\big\langle \Pi_j Cf, \Pi_j C' f\big\rangle\Big\|. \tag{22}$$

Also we have

$$
\begin{aligned}
\Big\|\sum_{j\in J}\langle \Pi_j Cf, \Pi_j C'f\rangle\Big\| &= \Big\|\sum_{j\in J}\langle \Pi_j (CC')^{\frac{1}{2}}f, \Pi_j (CC')^{\frac{1}{2}}f\rangle\Big\| \\
&= \Big\|\big\{\Pi_j (CC')^{\frac{1}{2}}f\big\}_{j\in J}\Big\|^2 \\
&\leq \Big\|\big\{\Lambda_j (CC')^{\frac{1}{2}}f\big\}_{j\in J}\Big\|^2 + \Big\|\big\{(\Lambda_j-\Pi_j) (CC')^{\frac{1}{2}}f\big\}_{j\in J}\Big\|^2 \\
&= \Big\|\sum_{j\in J}\langle (\Lambda_j-\Pi_j) (CC')^{\frac{1}{2}}f, (\Lambda_j-\Pi_j) (CC')^{\frac{1}{2}}f\rangle\Big\| \\
&\quad + \Big\|\sum_{j\in J}\langle \Lambda_j (CC')^{\frac{1}{2}}f, \Lambda_j (CC')^{\frac{1}{2}}f\rangle\Big\| \\
&= \Big\|\sum_{j\in J}\langle (\Lambda_j-\Pi_j) Cf, (\Lambda_j-\Pi_j) C'f\rangle\Big\| + \Big\|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\Big\| \\
&\leq M_1\Big\|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\Big\| + \Big\|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\Big\| \\
&= (1+M_1)\Big\|\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle\Big\| \\
&\leq (1+M_1)B_1\|f\|^2. \qquad (23)
\end{aligned}
$$

Therefore from (22) and (23), it is clear that $\{\Pi_j : j \in J\}$ is a (C, C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. □

Proposition 3.9. Let $\{\Lambda_j \in End_A^*(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ and $\{\Gamma_j \in End_A^*(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ be two (C, C')-controlled g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$ with bounds B_1 and B_2, respectively. Then the operator $L_{(C,C')} : \mathcal{H} \to \mathcal{H}$ given by

$$
L_{(C,C')}(f) = \sum_{j\in J} C'\Gamma_j^*\Lambda_j Cf \qquad (24)
$$

is well defined and bounded with $\|L_{(C,C')}\| \leq \sqrt{B_1B_2}$. Also its adjoint operator is $L^*_{(C,C')}(g) = \sum_{j\in J} C\Lambda_j^*\Gamma_j C'g$.

Proof. For any $f \in \mathcal{H}$ and $K \subset J$, we have

$$\begin{aligned}\Big\|\sum_{j\in K} C'\Gamma_j^*\Lambda_j Cf\Big\|^2 &= \sup_{g\in\mathcal{H},\|g\|=1}\Big\|\Big\langle \sum_{j\in K} C'\Gamma_j^*\Lambda_j Cf,\ g\Big\rangle\Big\|^2\\ &= \sup_{g\in\mathcal{H},\|g\|=1}\Big\|\sum_{j\in K}\Big\langle \Lambda_j Cf, \Gamma_j C'g\Big\rangle\Big\|^2\\ &\leq \sup_{g\in\mathcal{H},\|g\|=1}\Big\|\sum_{j\in K}\Big\langle \Lambda_j Cf, \Lambda_j Cf\Big\rangle\Big\|\Big\|\sum_{j\in K}\Big\langle \Gamma_j C'g, \Gamma_j C'g\Big\rangle\Big\|\\ &\leq \Big\|\sum_{j\in K}\Big\langle \Lambda_j Cf, \Lambda_j Cf\Big\rangle\Big\|\ B_2\\ &\leq B_1B_2\|f\|^2.\end{aligned}$$

Since K is arbitrary the series $\sum_{j\in J} C'\Gamma_j^*\Lambda_j Cf$ converges in $\mathcal{H}$, and

$$\|L_{(C,C')}\| = \Big\|\sum_{j\in K} C'\Gamma_j^*\Lambda_j Cf\Big\| \leq \sqrt{B_1B_2}.$$

Moreover, we see that

$$\begin{aligned}\langle L_{(C,C')}f, g\rangle = \Big\langle \sum_{j\in K} C'\Gamma_j^*\Lambda_j Cf,\ g\Big\rangle = \sum_{j\in K}\Big\langle C'\Gamma_j^*\Lambda_j Cf,\ g\Big\rangle &= \sum_{j\in K}\Big\langle f\ , C\Lambda_j^*\Gamma_j C'g\Big\rangle\\ &= \Big\langle f\ , \sum_{j\in K} C\Lambda_j^*\Gamma_j C'g\Big\rangle.\end{aligned}$$

Thus $L^*_{(C,C')}(g) = \sum_{j\in J} C\Lambda_j^*\Gamma_j C'g.$ □

Theorem 3.10. Let $\{\Lambda_j \in End_A^*(\mathcal{H},\mathcal{H}_j) : j \in J\}$ be a (C,C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$, and $\{\Gamma_j \in End_A^*(\mathcal{H},\mathcal{H}_j) : j \in J\}$ be a (C,C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$. Assume that C and C' commute with each other and commute with $\Gamma_j^*\Gamma_j$. If the operator $L_{(C,C')}$ defined in (24) is surjective then $\{\Gamma_j : j \in J\}$ is also a (C,C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$.

Proof. It is given that $\{\Lambda_j : j \in J\}$ is a (C,C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$. Then by Theorem 3.7, the operator $T_{(C,C')} : \bigoplus_{j\in J}\mathcal{H}_j \to \mathcal{H}$ given by

$$T_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J}(CC')^{\frac{1}{2}}\Lambda_j^* g_j,\ \ \forall \{g_j\}_{j\in J} \in \bigoplus_{j\in J}\mathcal{H}_j$$

is well defined and bounded operator. By (6) its adjoint operator $T^*_{(C,C')}: \mathcal{H} \to \bigoplus_{j\in J} \mathcal{H}_j$ is given by

$$T^*_{(C,C')}(f) = \left\{\Lambda_j (C'C)^{\frac{1}{2}} f\right\}_{j\in J},\ \forall f \in \mathcal{H}.$$

Since $\{\Gamma_j : j \in J\}$ is also a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$, again by Theorem 3.7, the operator $P_{(C,C')}: \bigoplus_{j\in J} \mathcal{H}_j \to \mathcal{H}$ given by

$$P_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J} (CC')^{\frac{1}{2}} \Gamma_j^* g_j,\ \forall \{g_j\}_{j\in J} \in l^2(\{\mathcal{H}_j\}_{j\in J})$$

is well defined and bounded operator. Again its adjoint operator is given by

$$P^*_{(C,C')}(f) = \left\{\Gamma_j (C'C)^{\frac{1}{2}} f\right\}_{j\in J},\ \forall f \in \mathcal{H}.$$

Hence for any $f \in \mathcal{H}$, the operator defined in (24) can be written as

$$L_{(C,C')}(f) = \sum_{j\in J} C'\Gamma_j^* \Lambda_j C f = P_{(C,C')} T^*_{(C,C')}(f).$$

Since $L_{(C,C')}$ is surjective then for any $f \in \mathcal{H}$, there exists $g \in \mathcal{H}$ such that $f = L_{(C,C')}(g) = P_{(C,C')} T^*_{(C,C')}(g)$, and $T^*_{(C,C')}(g) \in l^2(\{\mathcal{H}_j\}_{j\in J})$. This implies that $P_{(C,C')}$ is surjective. As a result of Lemma 3.2, we have $P^*_{(C,C')}$ is bounded below, that is there exists $m > 0$ such that

$$\begin{aligned}
&\left\langle P^*_{(C,C')} f, P^*_{(C,C')} f\right\rangle \geq m\langle f, f\rangle, \forall f \in \mathcal{H}\\
\Rightarrow\ &\left\langle P_{(C,C')} P^*_{(C,C')} f, f\right\rangle \geq m\langle f, f\rangle, \forall f \in \mathcal{H}\\
\Rightarrow\ &\left\langle \sum_{j\in J} (CC')^{\frac{1}{2}} \Gamma_j^* \Gamma_j (C'C)^{\frac{1}{2}} f,\ f\right\rangle \geq m\langle f, f\rangle,\ \forall f \in \mathcal{H}\\
\Rightarrow\ &\sum_{j\in J} \left\langle \Gamma_j C f, \Gamma_j C' f\right\rangle \geq m\langle f, f\rangle,\ \forall f \in \mathcal{H}.
\end{aligned}$$

Hence $\{\Gamma_j : j \in J\}$ is also a (C, C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}\}_{j\in J}$. □

Acknowledgment

The author is thankful to the anonymous referees for their valuable suggestions and comments which improved the presentation of the paper.

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Received : *August* 2021
Accepted : *September* 2021