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Some approximations for an equation in modular spaces

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Abstract. In this paper, we introduce and obtain the general solution of a new mixed type quadratic-cubic functional equation. We investigate the stability of such functional equations in the modular space X_ρ by applying Δ_2-condition and the Fatou property (in some results) in the modular function ρ.

1. Introduction

We say that an equation is *stable* in some class of functions if any function from that class, satisfying the equation approximately (in some sense), is near (in some way) to an exact solution of the equation. In 1940, Ulam [40] asked the question concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [15] for the linear functional equation of Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [37] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

The stability problem for quadratic functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1}$$

has been studied in normed spaces by Skof [39] with constant bound. Thereafter, Czerwik [13] proved the Hyers-Ulam stability of the quadratic functional equation with nonconstant bound; some different version of quadratic, quadratic-reciprocal functional equations and their stabilities with applications can be found in [4], [7] and [16].

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The cubic function $f(x) = ax^3$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{2}$$

Hence, the equation (2) is called a *cubic functional equation* and every solution of equation (2) is said to be a *cubic function*. The stability result of equation (2) was obtained by Jun and Kim [21] for the first time. After that, they [22] introduced the following cubic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y)$$

and they established the general solution and the Hyers-Ulam stability problem for it; see [3],[6], [8] and [20] for some results of various cubic functional equations and also their reciprocal with the generalized Hyers-Ulam stability. More result of miscellaneous functional equations can be found in [2], [5], [9], [10], [17], [35], [36], and references therein.

In [12], Chang and Jung introduced the following mixed type quadratic and cubic functional equation

$$6f(x+y) - 6f(x-y) + 4f(3y) = 3f(x+2y) - 3f(x-2y) + 9f(2y). \tag{3}$$

They established the general solution of functional equation (3) and investigated the Hyers-Ulam-Rassias stability of this equation; for a different form of mixed type quadratic-cubic functional equation, we refer to [23].

Nakano [31] initiated the study of the modular on linear spaces and the relevant theory of modular linear spaces as generalizations of metric spaces. Next, Luxemburg [26], Mazur, Musielak and Orlicz [28, 29, 30] thoroughly developed it extensively. Since then, the theory of modulars and modular spaces is widely applied in the study of interpolation theory [27, 25] and various Orlicz spaces [32]. A modular yields less properties than a norm does, but it makes a more sense in many special situations. When we work in a modular space, it is frequently assumed that the modular satisfies extra additional properties like some relaxed continuity or some Δ_2-condition. As for the mentioned condition, Khamsi [18] studied the stability of quasicontraction mappings in modular spaces without Δ_2-condition by using the fixed point theorem. The stability results of additive functional equations in modular spaces equipped with the Fatou property and Δ_2-condition were investigated by Sadeghi [38] who used Khamsi's fixed point theorem. In addition, the stability of quadratic functional equations in modular spaces satisfying the Fatou property without using the Δ_2-condition was investigated in [41]. Park et al., investigated the stability of additive and Jensen-additive functional equations without using the Δ_2-condition by a fixed point method [33]. An alternative generalized Hyers-Ulam stability theorem of a modified quadratic functional equation in a modular spaces using Δ_3-condition without the Fatou property on a modular function is in [19]. Furthermore, a refined stability result and alternative stability results for additive and quadratic functional equations using direct method in modular spaces are given in [24].

In this paper, we consider the mixed type quadratic-cubic functional which is somewhat different from (3) as follows:

$$f(x+2y)-f(x-2y)=2[f(x+y)-f(x-y)]+3f(2y)-12f(y) \tag{4}$$

It is easily verified that the function $f(x)=ax^2+bx^3$ is a solution of equation (4). The main purpose of the present paper is to solve and to prove the Hyers-Ulam stability problem equation (4) in the modular space X_ρ by applying Δ_2-condition and the Fatou property (in some results) in the modular function ρ.

2. Preliminary notations

In this section, we recall some basic facts concerning modular spaces and some preliminary results.

Definition 2.1. Let X be a linear space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized function $\rho : X \longrightarrow [0,\infty]$ is called a *modular* if it satisfies the following three conditions for elements $\alpha,\beta\in\mathbb{K}$, $x,y\in X$;

(i) $\rho(x)=0$ if and only if $x=0$;
(ii) $\rho(\alpha x)=\rho(x)$ for all scalar α with $|\alpha|=1$;
(iii) $\rho(\alpha x+\beta y)\le\rho(x)+\rho(y)$ for all scalar $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.

If the condition (iii) is replaced by $\rho(\alpha x+\beta y)\le\alpha^t\rho(x)+\beta^t\rho(y)$ when $\alpha^t+\beta^t=1$ and $\alpha,\beta\ge 0$ with an $t\in(0,1]$, then ρ is called an *t-convex modular*. 1-convex modulars are called *convex modulars*. For a modular ρ, there corresponds a linear subspace X_ρ of X, given by $X_\rho := \{x\in X:\ \rho(\lambda x)\to 0 \,\text{as}\, \lambda\to 0\}$. In this case X_ρ is called a *modular space*.

Let ρ be a convex modular. Then, the modular space X_ρ can be equipped with a norm called the *Luxemburg norm*, defined by $\|x\|_\rho = \inf\{\lambda>0:\ \rho\left(\frac{x}{\lambda}\right)\le 1\}$.

Here, we remind the following notes which are taken from [19].

(1) If ρ is a modular on X, then $\rho(tx)$ is an increasing function in $t\ge 0$ for each fixed $x\in X$, that is, $\rho(ax)\le\rho(bx)$ whenever $0\le a<b$;
(2) If ρ is a convex modular on X and $|\alpha|\le 1$, then $\rho(ax)\le|\alpha|\rho(x)$ for all $x\in X$. In particular, if $\alpha_j\ge 0\,(j=1,2,\ldots,n)$ with $0<\sum_{j=1}^n\alpha_j\le 1$, then $\rho\left(\sum_{j=1}^n\alpha_j x_j\right)\le\sum_{j=1}^n\alpha_j\rho(x_j)$ for all $x_j\in X$.

Definition 2.2. Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. Then

(i) $\{x_n\}$ is ρ-convergent to a point $x_*\in X_\rho$ and write $x_n\xrightarrow{\rho}x_*$ if $\rho(x_n-x_*)\to 0$ as $n\to\infty$);

(ii) $\{x_n\}$ is ρ-Cauchy sequence if for any $\epsilon > 0$ one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$;

(iii) A subset $Y \subseteq X_\rho$ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent to a point in Y.

Example 2.3. [34] Let $\psi : [0, \infty) \longrightarrow \mathbb{R}$ be a function defined by $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. If moreover ψ is convex, continuous and nondecreasing, then ψ is called an *Orlicz function*. For a measure space $(X, \sum, \mu)$, suppose that $L^0(\mu)$ is the set of all measurable functions on X. For each $f \in L^0(\mu)$, define $\rho_\psi(f) = \int_X \psi(|f|)d\mu$. Then, ρ_ψ is a modular and the corresponding modular space is called an *Orlicz space* and denoted by

$$L_\psi = \{f \in L^0(\mu) |\ \rho_\psi(\lambda f) \to 0 \,\text{as}\, \lambda \to 0\}.$$

One can check that L_ψ is ρ_ψ-complete.

A modular function ρ is said to satisfy the Δ_s-*condition* if there exists $\kappa > 0$ such that $\rho(sx) \le \kappa\rho(x)$ for all $x \in X_\rho$. Throughout this paper, we say that the constant κ is a Δ_s-constant related to Δ_s-condition. Suppose that ρ is convex and satisfies Δ_s-condition with Δ_s-constant κ. If $\kappa < s$, then $\rho(x) \le \kappa\rho\left(\frac{x}{s}\right) \le \frac{\kappa}{s}\rho(x)$, which implies $\rho = 0$. Hence, we must have the Δ_s-constant $\kappa \ge s$ if ρ is convex modular. It is said that the modular ρ has the Fatou property if and only if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x in the modular space X_ρ.

3. Stability of (4) in modular spaces

In this section, we prove the generalized Hyers-Ulam-Rassias stability of the mixed type quadratic-cubic functional equation (4). From now on, we assume that V is a real vector space and X_ρ is a complete modular space satisfies the Δ_2-condition and has the Fatou property unless otherwise stated explicitly.

Before proceeding the proof of our main results in this section, we shall bring the following lemma.

Lemma 3.1. *Let X and Y are real vector spaces. Suppose that $f : X \longrightarrow Y$ satisfies (4) for all $x, y \in X$.*

(i) *If f is even, then f is quadratic;*

(ii) *If f is odd, then f is cubic.*

Proof. (i) Putting $x = y = 0$ in (4), we have $f(0) = 0$. Letting $x = 0$ in (4), we get by the evenness of f that $f(2y) = 4f(y)$ for all $y \in X$. The last equality converts (4) to

$$f(x + 2y) - f(x - 2y) = 2[f(x + y) - f(x - y)] \tag{5}$$

for all $x, y \in X$. It is seen that (5) is the same relation (2.2) from [12]. Repeating the proof of Lemma 2.1 of [12], one can find (1) for f.

(ii) Putting $x = 0$ in (4) and using the oddness of f, we have $f(2y) = 8f(y)$ for all $y \in X$. Applying the last equality in (4), we arrive at

$$f(x + 2y) - f(x - 2y) = 2[f(x + y) - f(x - y)] + 12f(y) \tag{6}$$

for all $x, y \in X$. Replacing (x, y) by (y, x) in (6), we obtain

$$f(2x + y) + f(2x + y) = 2[f(x + y) + f(x - y)] + 12f(x) \tag{7}$$

for all $x, y \in X$. This completes the proof. $\square$

Similar to Theorem 2.3 of [**12**], we have the next result. Since the proof is the same, is omitted.

Theorem 3.2. *Let X and Y be real vector spaces. Then, a mapping $f : X \longrightarrow Y$ satisfies functional equation (4) for all $x, y \in X$ if and only if there exists a unique symmetric biadditive mapping $\mathcal{Q} : X \times X \longrightarrow Y$ and a unique mapping $\mathcal{C} : X \times X \times X \longrightarrow Y$ such that $f(x) = \mathcal{Q}(x, x) + \mathcal{C}(x, x, x)$ for all $x \in X$, and $\mathcal{C}$ is symmetric for fixed one variable and is additive for fixed two variables.*

Here and subsequently, given $f : X \longrightarrow Y$, for simplicity, we define the difference operator $\Lambda f : X \times X \longrightarrow Y$ by

$$\Lambda f(x, y) := f(x + 2y) - f(x - 2y) - 2[f(x + y) - f(x - y)] - 3f(2y) + 12f(y)$$

for all $x, y \in X$.

In the upcoming theorem, we prove the stability of the functional equation (4) as a quadratic functional equation (the even case of (4)) in the modular spaces.

Theorem 3.3. *Let $s \in \{1, -1\}$. Let $\phi : V \times V \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{2|s-1|j}}{4^j} \phi(2^{sj}x, 2^{sj}y) < \infty \tag{8}$$

for all $x, y \in V$. Suppose that $f : V \longrightarrow X_\rho$ is an even mapping satisfying $f(0) = 0$ (when $s = 1$) and the inequality

$$\rho(\Lambda f(x, y)) \leq \phi(x, y) \tag{9}$$

for all $x, y \in V$. Then, there exists a unique quadratic mapping $\mathcal{Q} : V \longrightarrow X_\rho$ such that

$$\rho(f(x) - \mathcal{Q}(x)) \leq \frac{1}{12} \sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{2|s-1|j} \phi(0, 2^{js}x)}{4^j} \tag{10}$$

for all $x \in V$.

Proof. We firstly consider $s=1$. Note that in this case $f(0)=0$ is assumed. Replacing (x,y) by $(0,x)$ in (9), we get

$$\rho(-3f(2x)+12f(x)) \leq \phi(0,x) \tag{11}$$

for all $x \in V$, and so

$$\rho(f(2x)-4f(x)) \leq \frac{1}{3}\phi(0,x) \tag{12}$$

for all $x \in V$. The inequality (12) implies that

$$\rho\left(f(x)-\frac{f(2x)}{4}\right) \leq \frac{1}{12}\phi(0,x)$$

for all $x \in V$. Once more, by induction on k, one can prove the following functional inequality

$$\rho\left(f(x)-\frac{f(2^k x)}{4^k}\right) \leq \frac{1}{12}\sum_{j=0}^{k-1}\frac{\phi(0,2^j x)}{4^j} \tag{13}$$

for all $x \in V$. Now, Interchanging x by $2^l x$ in (13), we have

$$\rho\left(\frac{f(2^l x)}{4^l}-\frac{f(2^{k+l}x)}{4^{k+l}}\right) \leq \frac{1}{12}\sum_{j=l}^{k+l-1}\frac{\phi(0,2^j x)}{4^j}$$

for all $x \in V$. Since the right-hand side of the above inequality tends to zero as l goes to infinity, the sequence $\{\frac{f(2^k x)}{4^k}\}$ is a ρ-Cauchy sequence in X_ρ and so the mentioned sequence is ρ-convergent on X_ρ. Thus, we may define the mapping $\mathcal{Q}: V \longrightarrow X_\rho$ via $\mathcal{Q}(x) = \rho - \lim_{k\to\infty}\frac{f(2^k x)}{4^k}$ for all $x \in V$. In other words, $\lim_{k\to\infty}\rho\left(\frac{f(2^k x)}{4^k}-\mathcal{Q}(x)\right)=0$. Replacing (x,y) by $(2^k x, 2^k y)$ in (9), and dividing the resulting inequality by 4^k, we get

$$\rho\left(\frac{1}{4^k}\Lambda f(2^k x,2^k y)\right) \leq \frac{1}{4^k}\rho\left(\Lambda f(2^k x,2^k y)\right) \leq \frac{1}{4^k}\phi(2^k x,2^k y)$$

for all $x,y \in V$. Similar to the proof of [19, Theorem 4], one can show that $\Lambda\mathcal{Q}(x,y)=0$ for all $x,y \in V$. This means that $\mathcal{Q}$ is a quadratic mapping. Now, it follows from the Fatou property of modular ρ that

$$\rho\left(f(x)-\mathcal{Q}(x)\right) \leq \liminf_{k\to\infty}\rho\left(f(x)-\frac{f(2^k x)}{4^k}\right) \leq \frac{1}{12}\sum_{j=0}^{k-1}\frac{\phi(0,2^j x)}{4^j}$$

for all $x \in V$, which shows that relation (10) holds. For the uniqueness of $\mathcal{Q}$, we assume that there exists another quadratic mapping $\mathcal{Q}_0: V \longrightarrow X_\rho$ such that

$$\rho(f(x)-\mathcal{Q}_0(x)) \leq \frac{1}{12}\sum_{j=0}^{\infty}\frac{\phi(0,2^j x)}{4^j}$$

for all $x \in V$ such that $\mathcal{Q}_0(x_*) \neq \mathcal{Q}(x_*)$ for some $x_* \in V$. In other words, there is a positive constant $\delta > 0$ such that $\rho(\mathcal{Q}_0(x_*) - \mathcal{Q}(x_*)) > \delta$. On the other hand, that there is a positive integer $p_0 \in \mathbb{N}$ such that $\frac{1}{12}\sum_{j=p_0}^{\infty}\frac{\phi(0,2^jx)}{4^j} < \delta$. Since $\mathcal{Q}$ and $\mathcal{Q}_0$ are quadratic mappings, we have $\mathcal{Q}_0(2^{p_0}x) = 4^{p_0}\mathcal{Q}_0(x)$ and $\mathcal{Q}(2^{p_0}x) = 4^{p_0}\mathcal{Q}(x)$. Hence,

$$\begin{aligned}\delta < \rho(\mathcal{Q}_0(x) - \mathcal{Q}(x)) &\leq \frac{1}{4^{p_0}}\rho\left(\mathcal{Q}_0(2^{p_0}x) - f(2^{p_0}x)\right) + \frac{1}{4^{p_0}}\rho\left(f(2^{p_0}x) - \mathcal{Q}(2^{p_0}x)\right)\\ &\leq \frac{1}{12}\sum_{j=0}^{\infty}\frac{\phi(0,2^{j+p_0}x)}{4^{j+p_0}} = \frac{1}{12}\sum_{j=p_0}^{\infty}\frac{\phi(0,2^jx)}{4^j} < \delta\end{aligned}$$

which is a contradiction. Now, assume that $s = -1$. Since $\kappa \geq 2$, (8) implies that $f(0) = 0$. It follows (12) that

$$\rho\left(f(x) - 4f\left(\frac{x}{2}\right)\right) \leq \frac{1}{3}\phi\left(0, \frac{x}{2}\right) \tag{14}$$

for all $x \in V$. By the convexity of the modular ρ, Δ_2-condition and (14), we have

$$\begin{aligned}\rho\left(f(x) - 4^2f\left(\frac{x}{2^2}\right)\right) &\leq \frac{1}{4}\rho\left(4f(x) - 4^2f\left(\frac{x}{2}\right)\right) + \frac{1}{4}\rho\left(4^2f\left(\frac{x}{2}\right) - 4^3f\left(\frac{x}{2^2}\right)\right)\\ &\leq \frac{1}{3}\left[\frac{\kappa^2}{4}\phi\left(0, \frac{x}{2}\right) + \frac{\kappa^4}{4}\phi\left(0, \frac{x}{2^2}\right)\right]\end{aligned}$$

for all $x \in V$. It is routine to show by induction on $k > 1$ that

$$\rho\left(f(x) - 4^kf\left(\frac{x}{2^k}\right)\right) \leq \frac{1}{3}\left[\sum_{j=1}^{k-1}\frac{\kappa^{2(2j-1)}}{4^j}\phi\left(0, \frac{x}{2^j}\right) + \frac{\kappa^{4(k-1)}}{4^{k-1}}\phi\left(0, \frac{x}{2^k}\right)\right] \tag{15}$$

for all $x \in V$. Replacing x by $\frac{x}{2^l}$ in (15), we get

$$\begin{aligned}\rho\left(4^lf\left(\frac{x}{2^l}\right) - 4^{k+l}f\left(\frac{x}{2^{k+l}}\right)\right) &\leq \kappa^{2l}\rho\left(f\left(\frac{x}{2^l}\right) - 4^kf\left(\frac{x}{2^{k+l}}\right)\right)\\ &\leq \frac{1}{3}\left[\kappa^{2l}\sum_{j=1}^{k-1}\frac{\kappa^{2(2j-1)}}{4^j}\phi\left(0, \frac{x}{2^{j+l}}\right) + \kappa^{2l}\frac{\kappa^{4(k-1)}}{4^{k-1}}\phi\left(0, \frac{x}{2^{k+l}}\right)\right]\\ &\leq \frac{1}{3}\left[\frac{4^l}{\kappa^{2l}}\sum_{j=1+1}^{k+l-1}\frac{\kappa^{2(2j-1)}}{4^j}\phi\left(0, \frac{x}{2^j}\right) + \frac{4^l}{\kappa^{2l}}\frac{\kappa^{4(k+l-1)}}{4^{k+l-1}}\phi\left(0, \frac{x}{2^{k+l}}\right)\right]\end{aligned} \tag{16}$$

for all $x \in V$. It follows from (8) and (16) that the sequence $\{4^kf\left(\frac{x}{2^k}\right)\}$ is a ρ-Cauchy sequence in X_ρ. Hence, there exists the mapping $\mathcal{Q} : V \longrightarrow X_\rho$ such that $\mathcal{Q}(x) = \rho - \lim_{k\to\infty} 4^kf\left(\frac{x}{2^k}\right)$. This means that the mentioned sequence is a ρ-convergent to $\mathcal{Q}(x)$. Using the Δ_2-condition without applying the Fatou property, we obtain

$$\begin{aligned}\rho\left(f(x)-\mathcal{Q}(x)\right) &\leq \frac{1}{4}\rho\left(4f(x)-4^{k+1}f\left(\frac{x}{2^k}\right)\right)+\frac{1}{4}\rho\left(4^{k+1}f\left(\frac{x}{2^k}\right)-4\mathcal{Q}(x)\right)\\ &\leq \frac{\kappa^2}{4}\rho\left(f(x)-4^kf\left(\frac{x}{2^k}\right)\right)+\frac{\kappa^2}{4}\rho\left(4^kf\left(\frac{x}{2^k}\right)-\mathcal{Q}(x)\right)\\ &\leq \frac{1}{3}\left[\frac{\kappa^2}{4}\sum_{j=1}^{k-1}\frac{\kappa^{2(2j-1)}}{4^j}\phi\left(0,\frac{x}{2^j}\right)+\frac{\kappa^2}{4}\frac{\kappa^{4(k-1)}}{4^{k-1}}\phi\left(0,\frac{x}{2^k}\right)\right]\\ &+\frac{\kappa^2}{12}\rho\left(4^kf\left(\frac{x}{2^k}\right)-\mathcal{Q}(x)\right)\end{aligned}$$

for all $x \in V$. Letting $k \to \infty$, we see that (10) holds. The rest of the proof is similar to the case $s = 1$. This completes the proof. $\square$

The following corollaries are the direct consequences of Theorem 3.3 concerning the stability of (4).

Corollary 3.4. *Given $\theta > 0$ and $r > 0$ such that $r \neq 2, \log_2^{\frac{\kappa^4}{4}}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ is an even mapping satisfying*

$$\rho(\Lambda f(x,y)) \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in V$, then there exists a unique quadratic mapping $\mathcal{Q} : V \longrightarrow X_\rho$ such that

$$\rho(f(x)-\mathcal{Q}(x)) \leq \begin{cases} \frac{\theta}{3(4-2^r)}\|x\|^r & r \in (0,2) \\ \\ \frac{\kappa^4\theta}{12(2^{r+2}-\kappa^4)}\|x\|^r & r \in (\log_2^{\frac{\kappa^4}{4}}, \infty) \end{cases}$$

for all $x \in V$.

Proof. We firstly note that $f(0) = 0$. Putting $\phi(x,y) = \theta(\|x\|^r + \|y\|^r)$ in Theorem 3.3, one can obtain the first and second inequalities for $s = 1$ and $s = -1$, respectively. $\square$

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1\varphi(a_1,\ldots,a_n) = \mathcal{F}_2\varphi(a_1,\ldots,a_n) \tag{17}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1\varphi_0(a_1,\ldots,a_n), \mathcal{F}_2\varphi_0(a_1,\ldots,a_n)) \leq \psi(a_1,\ldots,a_n), \qquad a_1,\ldots,a_n \in A,$$

fulfils (17); this definition is introduced in [11]. In other words, a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$.

Corollary 3.5. *Given $\theta, p, q > 0$ and $r = p + q$ such that $r \neq 2, \log_2^{\frac{\kappa^4}{4}}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ is an even mapping satisfying*

$$\rho(\Lambda_{m,n} f(x, y)) \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in V$, then f is quadratic.

Remark 3.6. We should remember that

(i) in the case $s = 1$ of Theorem 3.3, we have used the Fatou property while the Δ_2-condition is not applied and vice versa for the case $s = -1$;

(ii) in Corollary 3.4 and Corollary 3.5, if Δ_2-constant is $\kappa = 2$, then $\log_2^{\frac{\kappa^4}{4}} = 2$. Thus, the second conditions convert to $r \in (2, \infty)$.

We have the next result which is analogous to Theorem 3.3 for functional equation (4) in the odd case.

Theorem 3.7. *Let $s \in \{1, -1\}$. Let $\phi : V \times V \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3|s-1|j}}{8^j} \phi(2^{sj} x, 2^{sj} y) < \infty \tag{18}$$

for all $x, y \in V$. Suppose that $f : V \longrightarrow X_\rho$ is an odd mapping satisfying the inequality

$$\rho(\Lambda f(x, y)) \leq \phi(x, y) \tag{19}$$

for all $x, y \in V$. Then, there exists a unique cubic mapping $\mathcal{C} : V \longrightarrow X_\rho$ such that

$$\rho(f(x) - \mathcal{C}(x)) \leq \frac{1}{8} \sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3|s-1|j} \phi(0, 2^{js} x)}{8^j} \tag{20}$$

for all $x \in V$.

Proof. Replacing (x, y) by $(0, x)$ in (19) and using the oddness property, we have

$$\rho(f(2x) - 8f(x)) \leq \phi(0, x) \tag{21}$$

for all $x \in V$. For the rest of the proof, one can repeat the same process in the proof of Theorem 3.3 after relation (12) to obtain the desired result. □

The upcoming results are some consequences of Theorem 3.7 concerning the stability of (4) when f is an odd mapping. Since the proofs are similar to the proofs of Corollaries 3.4 and 3.5, we omit them.

Corollary 3.8. *Given $\theta > 0$ and $r \in \mathbb{R}$ such that $r \neq 3, \log_2^{\frac{\kappa^6}{8}}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ is an odd mapping satisfying*

$$\rho(\Lambda f(x, y)) \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in V$, then there exists a unique cubic mapping $\mathcal{C} : V \longrightarrow X_\rho$ such that

$$\rho(f(x) - \mathcal{C}(x)) \leq \begin{cases} \frac{\theta}{8-2^r}\|x\|^r & r \in (0, 3) \\ \\ \frac{\kappa^6\theta}{8(2^{r+3}-\kappa^6)}\|x\|^r & r \in (\log_2^{\frac{\kappa^6}{8}}, \infty) \end{cases}$$

for all $x \in V$.

Corollary 3.9. *Given $\theta, p, q > 0$ and $r = p+q$ such that $r \neq 3, \log_2^{\frac{\kappa^6}{8}}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ is an odd mapping satisfying*

$$\rho(\Lambda f(x, y)) \leq \theta\|x\|^p\|y\|^q$$

for all $x, y \in V$, then f is cubic.

Here, by using Theorems 3.3 and 3.7, we prove the generalized Hyers-Ulam-Rassias stability of the mixed type quadratic and cubic functional equation (4) when f is an arbitrary mapping.

Theorem 3.10. *Let $s \in \{1, -1\}$ and $t \in \{2, 3\}$. Let $\phi : V \times V \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{t|s-1|j}}{2^{tj}}\phi(2^{sj}x, 2^{sj}y) < \infty,$$

for all $x, y \in V$. Suppose that $f : V \longrightarrow X_\rho$ is a mapping satisfying $f(0) = 0$ (when $s = 1$) and the inequality

$$\rho(\Lambda f(x, y)) \leq \phi(x, y)$$

for all $x, y \in V$. Then, there exists a unique quadratic mapping $\mathcal{Q} : V \longrightarrow X_\rho$ and a unique cubic mapping $\mathcal{C} : V \longrightarrow X_\rho$ such that

$$\begin{aligned} &\rho(f(x) - \mathcal{Q}(x) - \mathcal{C}(x)) \\ &\leq \frac{1}{24}\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{2|s-1|j}\Phi(0, 2^{js}x)}{4^j} + \frac{1}{16}\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3|s-1|j}\Phi(0, 2^{js}x)}{8^j} \end{aligned} \tag{22}$$

for all $x \in V$, where

$$\Phi(x, y) = \frac{1}{2}[\phi(x, y) + \phi(-x, -y)]. \tag{23}$$

Proof. To find our purpose, we decompose f into the even part and odd part by setting

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \qquad f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad (x \in X).$$

We have $\rho(\Lambda f_o(x,y)) \leq \Phi(x,y)$ and $\rho(\Lambda f_e(x,y)) \leq \Phi(x,y)$ for all $x, y \in V$, where $\Phi(x,y)$ is given in (23). It follows from Theorem 3.3 that there exists a unique quadratic mapping $\mathcal{Q}_0 : V \longrightarrow X_\rho$ such that

$$\rho(f_e(x) - \mathcal{Q}_0(x)) \leq \frac{1}{12} \sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{2|s-1|j}\widetilde{\Phi}(0, 2^{js}x)}{4^j} \tag{24}$$

for all $x \in V$. Moreover, Theorem 3.7 implies that there exists a unique cubic mapping $\mathcal{C}_0 : X \longrightarrow Y$ such that

$$\rho(f_o(x) - \mathcal{C}_0(x)) \leq \frac{1}{8} \sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3|s-1|j}\widetilde{\Phi}(0, 2^{js}x)}{8^j} \tag{25}$$

for all $x \in V$. Now, by (24) and (25) we can obtain the inequality (22) where $\mathcal{Q}(x) = \frac{1}{2}\mathcal{Q}_0(x)$ and $\mathcal{C}(x) = \frac{1}{2}\mathcal{C}_0(x)$. □

The following corollaries are the direct consequences of Theorem 3.3 concerning the stability of (4).

Corollary 3.11. *Given $\theta > 0$ and $r > 0$ such that $r \neq t, \log_2^{\frac{\kappa^{2t}}{2^t}}$ when $t \in \{2,3\}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ is a mapping satisfying*

$$\rho(\Lambda f(x,y)) \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in V$, then there exists a unique quadratic mapping $\mathcal{Q} : V \longrightarrow X_\rho$ and a unique cubic mapping $\mathcal{C} : V \longrightarrow X_\rho$ such that

$$\rho(f(x) - \mathcal{Q}(x) - \mathcal{C}(x)) \leq \begin{cases} \left[\frac{2}{3(4-2^r)} + \frac{1}{2(8-2^r)})\right] \theta\kappa\|x\|^r & 0 < r < 2 \\ \left[\frac{\kappa^4}{24(2^{r+2}-\kappa^4)} + \frac{1}{2(8-2^r)}\right] \theta\kappa\|x\|^r & \log_2^{\frac{\kappa^4}{4}} < r < 3 \\ \left[\frac{\kappa^4}{24(2^{r+2}-\kappa^4)} + \frac{\kappa^6}{16(2^{r+3}-\kappa^6)}\right] \theta\kappa\|x\|^r & r > \log_2^{\frac{\kappa^6}{8}} \end{cases}$$

for all $x \in V$.

Corollary 3.12. *Given $\theta, p, q > 0$ and $r = p + q$ such that $r \neq t, \log_2^{\frac{\kappa^{2t}}{2^t}}$ when $t \in \{2, 3\}$. Let V be a normed space and X_ρ be a ρ-complete convex modular space. If $f : V \longrightarrow X_\rho$ be a mapping satisfying*

$$\rho(\Lambda f(x, y)) \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in V$, then f is qudratic-cubic.

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