Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 3, 2021, 33–38
doi: 10.30495/maca.2021.1932449.1015
ISSN 2716-9898

On Palais method in b-metric like spaces

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Abstract. This paper aims to prove that the Lipschitz constant in the Banach contraction principle belongs to the whole interval $[0, 1)$ for all the six classes of spaces viz. metric spaces, b-metric spaces, partial metric spaces, partial b-metric spaces, metric like space, and finally for more general spaces called b-metric like spaces. For the proof, the idea of Palais is used and applied in a more general setting. However, the current approach is a bit more general, because the presented result is applied to spaces, where the condition $d(x, y) = 0$ yields $x = y$ but not conversely. Accordingly, the outcome of the paper sums up, complements and binds together known results available in the current research literature.

1. Introduction

In 2007 Palais in [4] gave completely new and unexpectedly simple, short and beautiful proof of Banach's famous contraction principle. The announced idea in the aforementioned paper, motivated the authors of this article to implement a similar idea in other generalized metric spaces, including partial metric, metric like, b-metric, partial b-metric, and finally b-metric like spaces. By implementing Palais's idea for the more generalized b-metric like spaces, we succeed in this paper, by showing the Banach principle applies to the Lipschitz constant in the whole interval i.e., $k \in [0, 1)$. Earlier, and for a long time, many authors adhered to the interval $[0, \frac{1}{s})$ where s is a given constant in the definition of b-metric-like spaces with $s \geq 1$. For details, see, for example, a recent book [3]. So the present approach, using the method of Palais, we generalizes the Banach principle for all six classes of spaces and holds for the whole segment $[0, 1)$. It should be noted that we thus obtain a simple and beautiful proof that the appropriate Picard sequence is Cauchy throughout in each of the six types of generalized metric spaces discussed above.

As a matter of first importance, we present the following:

2010 *Mathematics Subject Classification.* Primary: 54H25; Secondary 47H10, 65F15.
Key words and phrases. Palais method, Banach contraction principle, fixed point
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Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping that satisfies the next for all x, y, z in X;

- (a) $d(x, y) = 0$ yields $x = y$;
- (b) $d(x, y) = 0$ if and only if $x = y$;
- (c) there exists $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$for all x, y, z in X;
- (d) $x = y$ yields $d(x, y) = 0$ that is., $d(x, x) = 0$;
- (e) $d(x, x) \leq d(x, y)$ for all x, y in X;
- (f) $x = y$ if and only if $d(x, x) = d(x, y) = d(y, y)$;
- (g) $d(x, y) \leq d(x, z) + d(z, y) - d(y, y)$;
- (h) $d(x, y) \leq s[d(x, z) + d(z, y)] - d(z, z)$

By using the properties (a)-(h), we obtain all six well known classes of metric spaces: metric spaces, partial metric spaces, metric-like spaces, b-metric spaces, partial b-metric spaces, and if (b) and (c) holds true, we say (X, d) is b-metric like space with the coefficient $s \geq 1$. In this case, if $s = 1$ then (X, d) is called metric space.

For $s = 1$ and (a) and (c) holds true then (X, d) is called metric like space.

The concept of b-metric-like spaces which generalizes the notions of partial metric spaces, metric-like spaces, and b-metric space was introduced by Alghamdi et al. in [1]. The relationship between these well-known interesting generalization is discussed as follows:

$$\begin{array}{ccccc} \text{Metric spaces} & \to & \text{Partial metric spaces} & \to & \text{Metric like-spaces} \\ \downarrow & & \downarrow & & \downarrow \\ b\text{-metric spaces} & \to & \text{Partial } b\text{-metric spaces} & \to & b\text{-metric like-spaces} \end{array}$$

Arrows stand for inclusions. The inverse inclusions do not hold. Metric spaces yield partial metric spaces; partial metric spaces yield metric like spaces. Furthermore, b-metric is partial b-metric, and partial b-metric is b-metric like space.
Also, metric is b-metric, partial is partial b-metric while metric like is b-metric like space. For partial metric, metric like, b-metric, partial b-metric and b-metric like we say that are generalizations of usual (standard) metric space.
To sum up the relationship between the underlying spaces, we formulate a flow diagram that gives us a better understanding of the relationship between the discussed generalized metric spaces.

Definitions for convergence and Cauchyness are formally the same as in case of metric spaces and b-metric spaces, while partial metric spaces, metric like spaces, partial b-metric spaces and b-metric like spaces do not adhere the same convergence and Cauchyness of metric spaces. However, generalized metric spaces (partial, metric like, partial b-metric and b-metric like case) adhere same Cauchyness and convergence (different from usual metric spaces). We give the corresponding definitions for convergence and Cauchy sequence only for b-metric like spaces, same follows for the rest of the generalized metric spaces discussed above.

Definition 1.1. Let $\{p_n\}$ be a sequence in a b-metric like space (X,d) with coefficient $s \geq 1$.Then

a) The sequence $\{p_n\}$ is said to be convergent to p if $\lim_{n\to+\infty} d(p_n,p) = d(p,p)$;
b) The sequence $\{p_n\}$ is said to be Cauchy in (X,d) if $\lim_{n,m\to+\infty} d(x_n,x_m)$ exists and is finite.
c) A b-metric like space (X,d) is said to be complete if for every Cauchy sequence $\{p_n\}$ in (X,d) there exists an element $p \in X$, such that

$$\lim_{n,m\to+\infty} d(p_n,p_m) = d(p,p) = \lim_{n\to+\infty} d(p_n,p).$$

d) For a mapping $T : (X,d) \to (X,d)$, we say that T is continuous in b-metric like context, if $\lim_{n\to+\infty} d(Tx_n,Tx) = d(Tx,Tx)$ whenever $\lim_{n\to+\infty} d(x_n,x) = 0 = d(x,y)$, where $\{x_n\}$ is a sequence in X and x is a point in X.

Next remark is an important criteria for the convergence of the sequences in b-metric like spaces:

Remark 1.1. [5, Remark 1] *It is worth to mention that in b-metric like space the limit of a sequences, is not unique and a convergent sequence need not be a Cauchy sequence (see [2], Example 7). However, if the b-metric like space (X,d) with the coefficient $s \geq 1$ is complete and the sequence $\{p_n\}$ is a Cauchy sequence with $\lim_{n,m\to+\infty} d(p_n,p_m) = 0$, then the limit of such sequence is unique. Indeed, in a such a case, if p_n tends p ($\lim_{n\to+\infty} d(p_n,p) = d(p,p)$) as n tends to infinity, we get that $d(p,p) = 0$. Now, if p_n tens p_1 and p_n tends p_2 where $p_1 \neq p_2$, we obtain $1/s \cdot d(p_1,p_2) \leq d(p_1,p_n) + d(p_n,p_2) \to d(p_1,p_1) + d(p_2,p_2) = 0+0 = 0$. From (a) it follows that $p_1 = p_2$, which is a contradiction. The same is true for partial metric, metric like and partial b-metric spaces.*

2. Main result

The aim of this section is to prove a Banac fixed point theorem, in the setting of b-metric like spaces. First, using the Palais method we obtain the following result.

Lemma 2.1. *Let (X,d) be a b-metric like space with $s \geq 1$ and $\{x_n\}$ be a sequence in X such that*

$$d(x_m,x_n) \leq kd(x_{m-1},x_{n-1}), \tag{1}$$

for all $m,n \in \mathbb{N}$, where $k \in [0,1)$. Then the following inequality holds

$$d(x_m,x_n) \leq \frac{s(k^m+k^n)d(x_0,x_{n_0})}{1-k^{n_0}s}, \tag{2}$$

where $n_0 \in \mathbb{N}$ such that $k^{n_0}s < 1$.

Proof. Let $n_0 \in \mathbb{N}$ such that $k^{n_0}s < 1$. Using condition (1), we obtain

$$d(x_{m-1+n_0}, x_{n-1+n_0}) \leq k^{n_0-1} d(x_m, x_n). \tag{3}$$

Using condition (c) from (3) we have

$$\begin{aligned} d(x_m, x_n) &\leq ks[d(x_{m-1}, x_{m-1+n_0}) + d(x_{m-1+n_0}, x_{n-1+n_0}) \\ &+ d(x_{n-1+n_0}, x_{n-1})] \\ &\leq ks[k^{m-1} d(x_0, x_{n_0}) + k^{n_0-1} d(x_m, x_n) + k^{n-1} d(x_{n_0}, x_0)]. \end{aligned}$$

So,

$$(1 - sk^{n_0}) d(x_m, x_n) \leq s(k^m + k^n) d(x_{n_0}, x_0), \tag{4}$$

that is, we get (2). □

Theorem 2.2. *Let (X, d) be a complete b-metric like space with $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq kd(x, y), \tag{5}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has an unique fixed point.

Proof. Firstly, for $k = 0$ the result is obvious. Therefore, we assume that $k \in (0, 1)$. Further, let $\{x_n = Tx_{n-1}\}$ be a Picard sequence based on any given point $x_0 \in X$. If $x_k = x_{k-1}$ for some $k \in \mathbb{N}$, we get that x_{k-1} is a unique fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Also, it is clear that if T has a fixed point then it is a unique. Now, we will prove the existence of fixed point of T. From (5) clearly follows that $d(x_m, x_n) \leq kd(x_{m-1}, x_{n-1})$. Now, using Lemma 2.1 we obtain that the sequence $\{x_n\}$ is a Cauchy in the framework of b-metric like space (X, d).

Now, according to the Definition 1.1 and Remark 1.1, there is a unique point $z \in X$ such that

$$\lim_{n,m\to+\infty} d(x_n, x_m) + \lim_{n\to+\infty} d(x_n, z) + d(z, z) = 0.$$

Since $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$ and $d(z, z) = 0$, it follows that $\lim_{n\to+\infty} d(x_n, z) = 0$. Further (1) implies that T is continuous in the following sense.

If $x_n \in X$ with $\lim_{n\to+\infty} d(x_n, x) = 0 = d(x, x)$ then $\lim_{n\to+\infty} d(Tx_n, Tx) = d(Tx, Tx)$. Now from our case, we get

$$\lim_{n\to+\infty} d(Tx_n, Tx) \leq k \cdot \lim_{n\to+\infty} d(x_n, x) = 0,$$

which further implies that $\lim_{n\to+\infty} d(Tx_n, Tx) = 0$.

Since,

$$d(Tx, Tx) \leq 2s \cdot d(Tx_n, Tx) \to 0 \text{ as } n \to +\infty,$$

it follows that

$$\lim_{n\to+\infty} d(Tx_n, Tx) = d(Tx, Tx) = 0.$$

That is

$$\lim_{n\to+\infty} d(x_{n+1}, Tx) = d(Tx, Tx) = 0.$$

Thus utilizing Remark 1.1, it follows that z is a unique fixed point of T.

□

3. Conclusions

We put forward a new simple and short proof of Banach's famous contraction principle in a more generalized setting of b-metric like spaces employing Palais's method. The presented result generalizes and extends some existing results found in literature furnishing that Lipschitz constant in the Banach contraction principle belongs to the whole interval $[0, 1)$ for all the six classes of spaces viz. metric spaces, b-metric spaces, partial metric spaces, partial b-metric spaces, metric like space, and finally for more general spaces viz. b-metric like spaces.

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Received : *June* 2021
Accepted : *July* 2021