Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 2, 2021, 68–74
doi: 10.30495/MACA.2021.1932071.1013
ISSN 2716-9898

Biflatness of Banach algebras modulo an ideal

Mohammad Ali Abolfathi[1,*] and Oluwatosin Temitope Mewomo[2]

Abstract. Biflatness of Banach algebras is one of the important topics in the study of cohomological properties of Banach algebras. This concept has a close relationship with the amenability of Banach algebras. In this paper, we introduce a new notion namely biflatness of Banach algebras modulo closed ideals. Moreover, we define the concept of virtual diagonal modulo ideals for investigating biflatness of Banach algebras modulo closed ideals. We show that biflatness of a Banach algebra A modulo I is equivalent to the existence of I-virtual diagonal modulo ideal I. By this result, we show that amenability of A/I implies biflatness of A modulo I. Moreover, we investigate the relationship of biflatness of the Banach algebra A modulo I with the biflatness of A/I. Finally, biflatness of Banach algebras modulo closed ideals is weaker than biprojectivity of them modulo closed ideals and provide examples to better understand the content

1. Introduction

Let A be a Banach algebra and X be a Banach A-bimodule. A map $D : A \longrightarrow X$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in A$. A derivation $D : A \longrightarrow X$ is called an inner derivation if for any $a \in A$, there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$. The space of all continuous derivations from A into X is denoted by $Z^1(A, X)$ and the space of all continuous inner derivations from A into X is denoted by $N^1(A, X)$. The first Hochschild cohomology group of A with coefficients in X is denoted by $\mathrm{H}^1(A, X) = Z^1(A, X)/N^1(A, X)$. A Banach algebra A is called amenable, if any continuous derivation from A into the dual of a Banach A-bimodule X is inner. In the other word, A is amenable if $\mathrm{H}^1(A, X^*) = 0$. As generalization of amenability of Banach algebras, Amini and Rahimi in [1] introduced the notion of amenability of Banach

2010 *Mathematics Subject Classification.* Primary: 43A07; Secondary: 46H25.
Key words and phrases. Amenability, amenability modulo an ideal, biflatness modulo an ideal, biprojectivity modulo and ideal
*Corresponding author.

algebras modulo an ideal. Let A be a Banach algebra and I be a closed ideal of A. Then A is amenable modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$ and every derivation D from A into X^* is inner on $A \backslash I = \{a \in A : a \notin I\}$. Moreover, for more information see [8].

Recently, Rahimi and Ghorbani introduced and studied the notion of biprojectivity of Banach algebras modulo an ideal in [5]. A Banach algebra A is called biprojective modulo I, if $\pi : A/I \widehat{\otimes} A/J \to A/I$ has a right inverse $\rho : A/I \to A/I \widehat{\otimes} A/J$ such that $\rho \circ \pi = id_{A/I}$, where J is the closed ideal that is generated by $\{ai : a \in A,\ i \in I\}$. In this paper, we denote the members of A/I and A/J by $\bar{a}$ and $\tilde{b}$, respectively.

An other important property of homology of Banach algebras is biflatness of these algebras. A Banach algebra A is called biflat if there exists an A-bimodule morphism $\lambda : A \to \left(A \widehat{\otimes} A\right)^{**}$ such that $\pi^{**} \circ \lambda$ is the natural embedding map in A^{**} [6]. Moreover, for the module version of biflatness of Banach algebras, we refer to [2]. In this paper, inspiring with [5], we define biflatness of Banach algebras modulo an ideal and give some results related to this new notion.

2. Biflatness modulo an ideal

Throughout of this section, we let A be a Banach algebra and I be a closed ideal of A. We commence with the following definition.

Definition 2.1. A Banach algebra A is biflat modulo I, if there is an A-bimodule morphis $\lambda : \frac{A}{I} \to \left(\frac{A}{I} \widehat{\otimes} \frac{A}{J}\right)^{**}$ such that $\pi^{**} \circ \lambda$ is the natural embedding map in $\left(\frac{A}{I}\right)^{**}$.

Definition 2.2. An element $M \in \left(\frac{A}{I} \widehat{\otimes} \frac{A}{J}\right)^{**}$ is called an I-virtual diagonal for A if for every $a \in A$ the following statements hold:

(i) $a \cdot M = M \cdot a$.
(ii) $a \cdot \pi^{**}(M) = \bar{a}$.

Lemma 2.1. *Let A be a Banach algebra and I be a closed ideal of A. If there is an I-virtual diagonal for A, then $\frac{A}{I}$ has a bounded approximate identity.*

Proof. Let M be an I-virtual diagonal for A. Thus there exists $(\bar{e}_\alpha)_{\alpha \in \Lambda} \subseteq \frac{A}{I}$ such that $\bar{e}_\alpha \xrightarrow{w^*} \pi^{**}(M)$. Then according to Definition 2.2(ii) we have:

$$\begin{aligned} a \cdot \pi^{**}(M) &= w^* - \lim_\alpha a\, \bar{e}_\alpha \\ &= \bar{a} \\ &= w^* - \lim_\alpha \bar{e}_\alpha\, a. \end{aligned}$$

This shows that $(\bar{e}_\alpha)_{\alpha \in \Lambda} \subseteq \frac{A}{I}$ is a bounded approximate identity for $\frac{A}{I}$. □

Theorem 2.2. *A be a Banach algebra and I be a closed ideal of A. Then A has an I-virtual diagonal if and only if $\frac{A}{I}$ has a bounded approximate identity and A is biflat modulo I.*

Proof. Let M be an I-virtual diagonal for A. Then by Lemma 2.1, $\pi^{**}(M)$ is a bounded approximate identity for $\frac{A}{I}$ where we show that by $(\bar{e}_\alpha)_{\alpha\in\Lambda}$. Clearly, π^{**} is an A-bimodule morphism. Define $\lambda : \frac{A}{I} \to \left(\frac{A}{I}\widehat{\otimes}\frac{A}{J}\right)^{**}$ by

$$\lambda(\bar{a}) = a \cdot M$$

for all $\bar{a} \in \frac{A}{I}$. We now show that λ is an A-bimodule morphism. For any $b \in A$ and $\bar{a} \in \frac{A}{I}$ we have:

$$\begin{aligned}\lambda(b\cdot\bar{a}) &= \lambda\left(\overline{ba}\right) = ba\cdot M\\ &= b\cdot(a\cdot M)\\ &= b\cdot\lambda(\bar{a})\end{aligned}$$

and

$$\begin{aligned}\lambda(\bar{a}\cdot b) &= \lambda\left(\overline{ab}\right) = ab\cdot M\\ &= a\cdot(b\cdot M) = a\cdot(M\cdot b)\\ &= \lambda(\bar{a})\cdot b\end{aligned}$$

Hence λ is an A-bimodule morphism. Moreover, for all $\bar{a} \in \frac{A}{I}$ we have:

$$\begin{aligned}\pi^{**}\circ\lambda(\bar{a}) &= \pi^{**}(a\cdot M)\\ &= a\cdot\pi^{**}(M)\\ &= a.\end{aligned}$$

Thus, A is biflat modulo I.

Conversely, suppose that $\frac{A}{I}$ has a bounded approximate identity say $(\bar{e}_\alpha)_{\alpha\in\Lambda}$ and A is biflat modulo I. Hence there is an A-morphism $\lambda : \frac{A}{I} \to \left(\frac{A}{I}\widehat{\otimes}\frac{A}{J}\right)^{**}$ such that $\pi^{**}\circ\lambda = \mathrm{id}_{\left(\frac{A}{I}\right)^{**}}$. Moreover, there exists $m \in \left(\frac{A}{I}\right)^{**}$ such that $\bar{e}_\alpha \xrightarrow{w^*} m$. We now define $M = m\otimes m$. Thus $M \in \left(\frac{A}{I}\widehat{\otimes}\frac{A}{J}\right)^{**}$. Let $\lambda(\bar{e}_\alpha) = M$. We have:

$$\begin{aligned}a\cdot\pi^{**}(M) &= \lim_\alpha a\cdot\pi^{**}(M)\\ &= \lim_\alpha a\cdot\pi^{**}\circ\lambda(\bar{e}_\alpha)\\ &= \lim_\alpha a\cdot\bar{e}_\alpha = \bar{a}\end{aligned}$$

and

$$\begin{aligned}a\cdot M &= \lim_\alpha a\cdot M\\ &= \lim_\alpha a\lambda(\bar{e}_\alpha)\\ &= \lambda(\lim_\alpha a\bar{e}_\alpha) = \lambda(a)\\ &= \lambda(\lim_\alpha \bar{e}_\alpha a)\\ &= \lambda(\lim_\alpha \bar{e}_\alpha)\cdot a\\ &= M\cdot a\end{aligned}$$

These show that M is an I-virtual diagonal for A. □

Corollary 2.3. *Let A be a Banach algebra and I be a closed ideal of A. If $\frac{A}{I}$ is amenable, then A is biflat modulo I and has an I-virtual diagonal for A.*

Proof. By [9, Theorem 2.2.4], $\frac{A}{I}$ has a virtual diagonal $M \in \left(\frac{A}{I}\widehat{\otimes}\frac{A}{I}\right)^{**}$ such that $\bar{a}\cdot\pi^{**}(M) = \bar{a}$ and $\bar{a}\cdot M = M\cdot\bar{a}$, for all $\bar{a} \in \frac{A}{I}$. Define a mapping $\lambda : \frac{A}{I} \to \left(\frac{A}{I}\widehat{\otimes}\frac{A}{J}\right)^{**}$ by $\lambda(\bar{a}) = \bar{a}\cdot M$, for all $\bar{a} \in \frac{A}{I}$. Similar to the proof of Theorem 2.2, λ is an A-bimodule morphism. Then

$$\begin{aligned}\pi^{**}\circ\lambda(\bar{a}) &= \pi^{**}(\bar{a}\cdot M)\\ &= \bar{a}\cdot\pi^{**}(M)\\ &= \bar{a}.\end{aligned}$$

Thus, A is biflat modulo I. Moreover, by Theorem 2.2, A has an I-virtual diagonal. □

Corollary 2.4. *Let A be a Banach algebra and I be a closed ideal of A. If A is amenable modulo I, then it is biflat modulo I.*

Proof. By [1, Theorem 6], $\frac{A}{I}$ is amenable and so by Corollary 2.3, A is biflat modulo I. □

Theorem 2.5. *Let A be a Banach algebra and I be a closed ideal of A. If A is biflat modulo I such that the mapping $\lambda : \frac{A}{I} \to \left(\frac{A}{I}\hat{\otimes}\frac{A}{J}\right)^{**}$ (defined in Definition 2.1) is an $\frac{A}{I}$-bimodule morphism and satisfies $\lambda\left(\frac{A}{I}\right) \subseteq \left(\frac{A}{I}\right)^{**}\widehat{\otimes}\left(\frac{A}{J}\right)^{**}$, then $\frac{A}{I}$ is biflat.*

Proof. Set $\lambda(\bar{a}) = \sum_{n=1}^{\infty}\bar{\alpha}_n\otimes\tilde{\beta}_n$, for all $\bar{a} \in \frac{A}{I}$, where $\bar{\alpha}_n \in \left(\frac{A}{I}\right)^{**}$ and $\tilde{\beta}_n \in \left(\frac{A}{J}\right)^{**}$. Consider a mapping $\iota : \frac{A}{J} \to \frac{A}{I}$ defined by $\iota(\tilde{a}) = \bar{a}$, for every $\tilde{a} \in \frac{A}{J}$. We now define $\Lambda := id_{\left(\frac{A}{I}\right)^{**}}\otimes\iota^{**}$. Thus, $\Lambda(\bar{\alpha}\otimes\tilde{\beta}) = \bar{\alpha}\otimes\bar{\beta}$, for all $\bar{\alpha} \in \left(\frac{A}{I}\right)^{**}$ and $\tilde{\beta} \in \left(\frac{A}{J}\right)^{**}$. Consider a mapping $\Phi : \frac{A}{I} \to \left(\frac{A}{I}\hat{\otimes}\frac{A}{J}\right)^{**}$ by $\Phi := \Lambda\circ\lambda$. Since Λ and λ are $\frac{A}{I}$-bimodule morphism, Φ is an $\frac{A}{I}$-bimodule morphism and

$$\begin{aligned}\pi^{**}_{\frac{A}{I}}\circ\Phi(\bar{a}) &= \pi^{**}_{\frac{A}{I}}\circ\Lambda\circ\lambda(\bar{a})\\ &= \pi^{**}_{\frac{A}{I}}\circ\Lambda\left(\sum_{n=1}^{\infty}\bar{\alpha}_n\otimes\tilde{\beta}_n\right)\\ &= \pi^{**}_{\frac{A}{I}}\left(\sum_{n=1}^{\infty}\bar{\alpha}_n\otimes\bar{\beta}_n\right)\\ &= \sum_{n=1}^{\infty}\pi^{**}_{\frac{A}{I}}\left(\bar{\alpha}_n\otimes\bar{\beta}_n\right)\\ &= \sum_{n=1}^{\infty}\bar{\alpha}_n\bar{\beta}_n = \sum_{n=1}^{\infty}\overline{\alpha_n\beta_n}\\ &= \sum_{n=1}^{\infty}\pi^{**}\left(\bar{\alpha}_n\otimes\tilde{\beta}_n\right)\\ &= \pi^{**}\left(\sum_{n=1}^{\infty}\bar{\alpha}_n\otimes\tilde{\beta}_n\right)\\ &= \pi^{**}\circ\lambda(\bar{a})\\ &= \bar{a},\end{aligned}$$

for every $\bar{a} \in \frac{A}{I}$. This completes the proof. □

We now give a result related to biprojectivity modulo an ideal of a Banach algebra and biflatness of that Banach algebra modulo that ideal.

Theorem 2.6. *Let A be a Banach algebra and I be a closed ideal of A. If A^{**} is biprojective modulo I^{**}, then A is biflat modulo I.*

Proof. Consider an A^{**}-bimodule morphism $\rho : \left(\frac{A}{I}\right)^{**} \to \left(\frac{A}{I}\right)^{**} \widehat{\otimes} \left(\frac{A}{J}\right)^{**}$ such that for $\pi_{\frac{A^{**}}{I^{**}}} : \frac{A^{**}}{I^{**}} \widehat{\otimes} \frac{A^{**}}{J^{**}} \to \frac{A^{**}}{I^{**}}$ we have $\pi_{\frac{A^{**}}{I^{**}}} \circ \rho = id_{\frac{A^{**}}{I^{**}}}$. Consider a mapping $\rho_0 = \rho|_{\frac{A}{I}} : \frac{A}{I} \to \left(\frac{A}{I}\right)^{**} \widehat{\otimes} \left(\frac{A}{J}\right)^{**}$. By [4, Lemma 7-1], there is a mapping $\Phi : \left(\frac{A}{I}\right)^{**} \widehat{\otimes} \left(\frac{A}{J}\right)^{**} \to \left(\frac{A}{I} \widehat{\otimes} \frac{A}{J}\right)^{**}$ such that for all $\bar{a} \in \frac{A}{I}$, $\tilde{b} \in \frac{A}{J}$ and $M \in \frac{A^{**}}{I^{**}} \widehat{\otimes} \frac{A^{**}}{J^{**}}$:

(i) $\Phi\left(\bar{a} \otimes \tilde{b}\right) = \bar{a} \otimes \tilde{b}$.
(ii) $\Phi(\bar{a} \cdot M) = \bar{a} \cdot \Phi(M)$ and $\Phi(M \cdot \bar{a}) = \Phi(M) \cdot \bar{a}$.
(iii) $\pi^{**}_{\frac{A}{I}}(\Phi(M)) = \pi_{\frac{A^{**}}{I^{**}}}(M)$.

$$\begin{aligned}
\pi^{**}_{\frac{A}{I}} \circ \Phi \circ \rho_0(\bar{a}) &= \pi^{**}_{\frac{A}{I}} \circ \Phi\left(\textstyle\sum_{n=1}^{\infty}\left(\bar{\alpha}_n \otimes \tilde{\beta}_n\right)\right) \\
&= \pi^{**}_{\frac{A}{I}}\left(\textstyle\sum_{n=1}^{\infty}\left(\bar{\alpha}_n \otimes \tilde{\beta}_n\right)\right) \\
&= \textstyle\sum_{n=1}^{\infty} \pi^{**}_{\frac{A}{I}}\left(\bar{\alpha}_n \otimes \tilde{\beta}_n\right) \\
&= \textstyle\sum_{n=1}^{\infty} \bar{\alpha}_n \tilde{\beta}_n = \sum_{n=1}^{\infty} \overline{\alpha_n \beta_n} \\
&= \textstyle\sum_{n=1}^{\infty} \pi_{\frac{A^{**}}{I^{**}}}\left(\bar{\alpha}_n \otimes \tilde{\beta}_n\right) \\
&= \pi_{\frac{A^{**}}{I^{**}}}\left(\textstyle\sum_{n=1}^{\infty}\left(\bar{\alpha}_n \otimes \tilde{\beta}_n\right)\right) \\
&= \pi_{\frac{A^{**}}{I^{**}}} \circ \rho_0(\bar{a}) \\
&= \bar{a}.
\end{aligned}$$

Thus, A is biflat modulo I. □

Example 2.3. Let $\mathcal{A}$ and $\mathfrak{A}$ be two Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule. The amalgamated Banach algebra $\mathcal{A} \rtimes \mathfrak{A}$ is defined by the following product and norm:

$$(a, \alpha)(b, \beta) = (ab + \alpha \cdot b + a \cdot \beta, \alpha\beta)$$

and

$$\| (a, \alpha) \|_1 = \| a \|_{\mathcal{A}} + \| \alpha \|_{\mathfrak{A}} .$$

These Banach algebras are introduced by Javanshiri and Nemati in [7] and their biprojectivity and biflatness are investigated by Ebadian and Jabbari in [3]. If $\mathcal{A} \rtimes \mathfrak{A}$ is amenable, then by [3, Proposition 4.3], $\mathfrak{A}$ is amenable. Moreover, from $\frac{\mathcal{A} \rtimes \mathfrak{A}}{\mathcal{A}} \cong \mathfrak{A}$ [7, Remark 3.1], so $\frac{\mathcal{A} \rtimes \mathfrak{A}}{\mathcal{A}}$ is amenable. Then by Corollary 2.3, $\mathcal{A} \rtimes \mathfrak{A}$ is biflat modulo $\mathcal{A}$.

3. Applications on semigroup algebras

In this section we consider discrete semigroups and Banach algebras on them. Let S be a semigroup and ρ be a group congruence on S. Define the quotient mapping $\varphi : S \longrightarrow \frac{S}{\rho}$. Then one can extend φ to a surjective homomorphism

$\overline{\varphi} : \ell^1(S) \longrightarrow \ell^1\left(\frac{S}{\rho}\right)$ where $\ker \overline{\varphi}$ is a closed ideal of $\ell^1(S)$ that we denoted it by I_ρ. Moreover, by [**1**, Lemma 15], $\ell^1\left(\frac{S}{\rho}\right) \cong \frac{\ell^1(S)}{I_\rho}$.

Theorem 3.1. *Let S be a semigroup and ρ be a group congruence on S such that $\frac{S}{\rho}$ is infinite and $\ell^1\left(\frac{S}{\rho}\right)$ is amenable. Then $\ell^1(S)$ is biflat modulo I_ρ and is not biprojective modulo I_ρ.*

Proof. Since $\ell^1\left(\frac{S}{\rho}\right)$ is amenable, $\frac{\ell^1(S)}{I_\rho}$ is amenable. Then by Corollary 2.3, $\ell^1(S)$ is biflat modulo I_ρ.

If $\ell^1(S)$ is biprojective modulo I_ρ, then by [**5**, Theorem 3.1], $\frac{S}{\rho}$ is finite. A contradiction, because $\frac{S}{\rho}$ was infinite. □

Example 3.1. Let $S = \{p^m q^n : m, n \geq 0\}$ be the bicyclic semigroup generated by p and q. By [**1**, Example 1], the idempotent set of S i.e., $E(S)$ is as follows:

$$\{p^n q^n : n = 0, 1, 2, ...\}.$$

Define the group congruence ρ on S by $s\rho t$ if and only if there exists $e \in E(S)$ such that $es = et$, for all $s, t \in S$. Then $\frac{S}{\rho} = \mathbb{Z}$. Hence, $\ell^1(\mathbb{Z}) = \frac{\ell^1(S)}{I_\rho}$. By [**1**, Example 1], $\ell^1(S)$ is amenable modulo I_ρ. Then by Theorem 3.1, $\ell^1(S)$ is biflat modulo I_ρ. Note that $\ell^1(S)$ is not biprojective modulo I_ρ, see [**5**, Example 3.3].

Proposition 3.2. *Let S be an amenable semigroup and ρ be a group congruence on S such that $\ker \rho$ is central and I_ρ has a bounded approximate identity. Then $\ell^1(S)$ is biflat modulo I_ρ.*

Proof. By [**1**, Theorem 3], $\ell^1(S)$ is amenable modulo I_ρ. Then by Theorem 2.5, $\ell^1(S)$ is biflat modulo I_ρ. □

References

[1] M. Amini and H. Rahimi, *Amenability of semigroups and their algebras modulo a group congruence*, Acta Math. Hung., **144**(2)(2014), 407-415.

[2] A. Bodaghi and M. Amini, *Module biprojective and module biflat Banach algebras*, U. P. B. Sci. Bull., Series A., **75**(3)(2013), 25-36.

[3] A. Ebadian and A. Jabbari, *Biprojectivity and biflatness of amalgamated duplication of Banach algebras*, J. Algebra Appl., **19**(7) (2020) 2050132.

[4] F. Ghahramani, R. J. Loy and G. A. Willis, *Amenability and weak amenability of second conjugate Banach algebras*, Proc. Amer. J. Math. Soc., **124**(1996) 1489-1497.

[5] M. Ghorbani and H. R. Rahimi, *Biprojectivty of Banach algebras modulo an ideal*, J. New Res. Math., **5**(18)(2019), 21-30.

[6] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, Dordrecht, 1989.

[7] H. Javanshiri and M. Nemati, *Amalgamated duplication of the Banach algebra $\mathfrak{A}$ along a $\mathfrak{A}$-bimodule $\mathcal{A}$*, J. Algebra Appl., **17**(9)(2018), 1850169.

[8] H. Rahimi and E. Tahmasebi, *Amenability and contractibility modulo an ideal of Banach algebras*, Abst. Appl. Anal., **2014**(2014).

[9] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics, Vol. 1774, Springer-Verlag, Berlin, 2002.

[1]Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran

m.abolfathi@urmia.ac.ir

[2]School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban, South Africa

mewomoo@ukzn.ac.za

Received: April 2021
Accepted: June 2021