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The coincidence point results and rational contractions in $E(s)$-distance spaces

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Abstract. The purpose of this article is to clarify the concept of semi-interior points of positive cones by presenting some results and examples in this context. Moreover, the new concept of $E(s)$-distance spaces is defined, which generalizes E-metric spaces. In addition, some coincidence point results have been obtained that extend and generalize some known results in the literature.

1. Introduction

Over the past several decades, fixed point theory in ordered normed spaces played an important role in optimization theory, game theory, variational inequalities, dynamical systems, fractals, graph theory, models in economy, computer science and many other fields. In 2007, Huang and Zhang [**9**] introduced the concept of cone metric space, in which convergent and Cauchy sequences were defined in terms of interior points in the ordered Banach space (see also [**1**]). In 2012, Rawashdeh et al. [**3**] used ordered normed space as an alternative of Banach space and introduced the concept of $\mathtt{E}$-metric space, also see on example [**2**]. In most of the results, $\mathtt{E}$ is considered as a Banach space with a solid cone, only few results in literature could be found in which the non-solid cones were considered [**13, 14**]. Kunze [**14**] used the concept of quasi-interior points of P in non solid cones.

Recently in 2017, the concept of semi-interior points was defined by Polyrakis [**16**]. According to Proposition 3.2 of [**4**] any semi-interior point of $\mathtt{E}^+$ is also an interior point of $\mathtt{E}^+$ with respect to another norm $|||.|||$ of $\mathtt{E}$ which coincides with the initial norm $\|.\|$ of $\mathtt{E}$ in $\mathtt{E}^+$. The class of cones with semi-interior point and empty interior is a class of cones larger than the one with nonempty interior as the examples of [**4**]. It is worth noting that fixed points results for ordered normed spaces can also

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proved for this larger class of cones with semi-interior points. In 2019, Mehmood et al. [15] proved some fixed point results in the frame of $\mathtt{E}$-metric spaces by inserting non-solid cones.

In this article we define $\mathtt{E}(\mathtt{s})$-distance space and generalize the results of [15] in the settings of $\mathtt{E}(\mathtt{s})$-distance space with non-solid and non-normal cones. Also, some coincidence point results have been obtained that extend and generalize some known results in the literature.

2. The $\mathtt{E}$-metric space

Throughout in this article, let $\mathtt{E}$ be a ordered normed space with a norm $\|.\|$ and $\mathtt{E}^+$ be a positive cone, such that for $\varkappa, \eta \in \mathtt{E}$, $\varkappa \preceq \eta$ if and only if $\eta - \varkappa \in \mathtt{E}^+$. The notion $\varkappa \prec \eta$ means that $\varkappa \preceq \eta$ and $\varkappa \neq \eta$, while $\varkappa \ll \eta$ stands for $\eta - \varkappa \in int(\mathtt{E}^+)$.

We start with the basics of ordered normed spaces and $\mathtt{E}$-metric spaces.

Definition 2.1. [3, 15] A vector space $\mathtt{E}$ over the set of real numbers, with a partial order relation $\preceq$ called is an ordered space if it satisfies these axioms

(1) for all $\varkappa, \eta$ and $\vartheta \in \mathtt{E}$, $\varkappa \preceq \eta$ implies $\varkappa + \vartheta \preceq \eta + \vartheta$,

(2) for all $a \in \mathbb{R}^+$ and $\varkappa \in \mathtt{E}$, $0_{\mathtt{E}} \preceq \varkappa$ implies $0_{\mathtt{E}} \preceq a\varkappa$. e Moreover, if $\mathtt{E}$ is equipped with a norm $\|.\|$, then $\mathtt{E}$ is called a normed ordered space.

Now we include the definition of $\mathtt{E}$-metric space defined in [3].

Definition 2.2. Let $\mathcal{M}$ be a nonempty set and $\mathtt{E}$ an ordered space, over the real scalars. An ordered $\mathtt{E}$-metric on $\mathcal{M}$ is a function $\mathtt{d}^{\mathtt{E}} : \mathcal{M} \times \mathcal{M} \to \mathtt{E}$ such that for all $\varkappa, \eta$ and $\vartheta \in \mathcal{M}$,

(1) $\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) \succeq 0_{\mathtt{E}}$ and $\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) = 0_{\mathtt{E}}$ if and only if $\varkappa = \eta$,

(2) $\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) = \mathtt{d}^{\mathtt{E}}(\eta, \varkappa)$,

(3) $\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) \preceq \mathtt{d}^{\mathtt{E}}(\varkappa, \vartheta) + \mathtt{d}^{\mathtt{E}}(\vartheta, \eta)$.

Then the pair $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ is called $\mathtt{E}$-metric space.

Remark 2.1. *In an $\mathtt{E}$-metric space, $\mathtt{E}^+$ is a closed* [3, Proposition 2.6]*. Also, set $int(\mathtt{E}^+)$ can be empty* [3, Example 2.8].

3. The positive cones and semi-interior points

In this section, we present the notion of semi-interior points of the positive cone $\mathtt{E}^+$ of an ordered norm space $\mathtt{E}$. Non-trivial examples for cones with empty interior but having semi-interior points have been discussed in [4, 15]. Here we provide some examples and remarks to elaborate this concept. The concept of semi-interior points have been defined by Polyrakis [16] and have been announced in Paris 2014 during the XXII European Workshop on General Equilibrium Theory, in his discussion on "Cones with semi-interior points and second welfare theorem".

Definition 3.1. Let $\mathtt{E}$ be an ordered normed space then any subset $\mathtt{E}^+$ of $\mathtt{E}$ is called positive cone if it satisfies these axioms:
(1) $a, b \in \mathbb{R}, a, b \geq 0, \varkappa, \eta \in \mathtt{E}^+$ implies $a\varkappa + b\eta \in \mathtt{E}^+$,
(2) if $\varkappa \in \mathtt{E}^+$ and $-\varkappa \in \mathtt{E}^+$ then $\varkappa = 0_{\mathtt{E}}$.

Definition 3.2. [4] The point $\varkappa_0 \in \mathtt{E}^+$ is a semi-interior point of $\mathtt{E}^+$ if there exists a real number $\rho > 0$ such that

$$\varkappa_0 - \rho U_+ \subseteq \mathtt{E}^+,$$

where U_+ is a positive part of unit ball and is defined as a common part of U and $\mathtt{E}^+$, where

$$U = \{\varkappa \in \mathtt{E} : \|\varkappa\| \leq 1\}.$$

Clearly any interior point of $\mathtt{E}^+$ is a semi-interior point. The set of semi-interior points of $\mathtt{E}^+$ is denoted by $(\mathtt{E}^+)^{\ominus}$, for $\varkappa, \eta \in \mathtt{E}^+$, $\varkappa \lll \eta$ if and only if $\eta - \varkappa \in (\mathtt{E}^+)^{\ominus}$.

Definition 3.3. [15]Let $\mathtt{E}$ be an ordered space with $(\mathtt{E}^+)^{\ominus}$ is nonempty and $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ be an $\mathtt{E}$-metric space. Let $(\varkappa_\nu)$ be a sequence in $\mathcal{M}$ and $\varkappa \in \mathcal{M}$. Then

- (i) $(\varkappa_\nu)$ $\mathfrak{e}$-converges to $\varkappa$ whenever for every $\mathfrak{e} \ggg 0_{\mathtt{E}}$, there exists $k \in \mathbb{N}$ such that $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa) \lll \mathfrak{e}$ for all $\nu \geq k$;
- (ii) $(\varkappa_\nu)$ is $\mathfrak{e}$-Cauchy whenever for every $\mathfrak{e} \ggg 0_{\mathtt{E}}$, there exists a natural number k such that $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu) \lll \mathfrak{e}$ for all $\nu, \mu \geq k$;
- (iii) $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ is $\mathfrak{e}$-complete if every $\mathfrak{e}$-Cauchy sequence is $\mathfrak{e}$-convergent.

We give some examples of positive cones with empty interior but having semi-interior points.

Example 3.4. In $\mathbb{R}^3$, consider a subspace $\mathtt{E} = \{(\varkappa, -\varkappa, \varkappa) : \text{ for all } \varkappa \in \mathbb{R}\}$. Then $\mathtt{E}^+ = \{(0,0,0)\}$ by using the ordering property

$$(\varkappa_1, \varkappa_2, \varkappa_3) \leq (\eta_1, \eta_2, \eta_3) \text{ if and only if } \varkappa_i \leq \eta_i \text{ for } i = 1, 2, 3,$$

of $\mathbb{R}^3$. Clearly $int(\mathtt{E}^+) = \phi$. Because zero element of $\mathtt{E}$ cannot be in $int(\mathtt{E}^+)$. Now for semi-interior points, let $\varkappa_0 = (0,0,0) \in \mathtt{E}^+$, clearly $\varkappa_0 \in U$, where

$$U = \{\varkappa \in \mathtt{E} : \|\varkappa\| \leq 1\}.$$

So $\varkappa_0$ belongs to the intersection of U_+ and $\mathtt{E}^+$. Hence

$$\varkappa_0 - \rho U_+ \subseteq \mathtt{E}^+.$$

Which means $\varkappa_0$ is a semi-interior point of $\mathtt{E}^+$. Thus we have $int(\mathtt{E}^+) = \emptyset$ and $(\mathtt{E}^+)^{\ominus} \neq \emptyset$.

Example 3.5. Consider a subspace, $\mathtt{E} = \{(\varkappa, \eta, \vartheta) \in \mathbb{R}^3 : \varkappa + \eta + \vartheta = 0\}$ of $\mathbb{R}^3$, then $\mathtt{E}^+ = \{(0,0,0)\}$. By using the same arguments as of above example we can show that $int(\mathtt{E}^+) = \emptyset$ and $(\mathtt{E}^+)^{\ominus} = \{(0,0,0)\}$.

Now, let us compare positive cone $\mathtt{E}^+$ and cone P defined in [9].

Remark 3.1. (1) *A positive cone can be zero, i.e.,* $\mathtt{E}^+ = \{0_\mathtt{E}\}$ *as shown in the above examples, whereas cone* $P \neq \{0\}$.
(2) *Positive cone is not necessarily a closed set.*

Example 3.6. Let $\mathtt{E} = \mathbb{R}^2$ and $\mathtt{E}^+ = \{(\varkappa_1, \varkappa_2) \in \mathbb{R}^2 : \varkappa_1 > 0\} \cup \{(0,0)\}$ is a positive cone but it is not closed.

Remark 3.2. *From above remarks we can conclude that a positive cone is not necessarily a cone defined in* [9].

Remark 3.3. *Let* $\mathtt{E}$ *be an ordered space having a norm and* $\mathtt{E}^+$ *be a set of positive elements of* $\mathtt{E}$*, then the zero element of* $\mathtt{E}^+$ *is neither in* $int(\mathtt{E}^+)$ *nor in* $(\mathtt{E}^+)^\ominus$. *However it belongs to* $(\mathtt{E}^+)^\ominus$ *only when* $\mathtt{E}^+ = \{0_\mathtt{E}\}$. *This fact can be derived by using the definition of semi-interior point. If* $\varkappa_0 = 0_\mathtt{E}$ *then* $-\rho U_+ \subseteq \mathtt{E}^+$*, which is not possible as* $\mathtt{E}^+$ *is a set of positive elements of* $\mathtt{E}$*. So* $0_\mathtt{E} \notin (\mathtt{E}^+)^\ominus$ *when* $\mathtt{E}^+ \neq \{0_\mathtt{E}\}$.

Proposition 3.4. *Let* $(\mathcal{M}, \mathtt{d}^\mathtt{E})$ *be an* $\mathtt{E}$*-metric space and* $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ *be an* $\mathfrak{e}$*-convergent sequence in* $\mathcal{M}$ *then* $\mathtt{E}^+$ *is not a normal cone provided that limit of* $\mathfrak{e}$*-convergent sequence is not unique.*

Proof. Assume that limit of convergent sequence is not unique. We argue by contradiction that $\mathtt{E}^+$ is a normal cone with normal constant M. Start with $\epsilon > 0$ and let $\mathfrak{e} \in (\mathtt{E}^+)^\ominus$ with $\|\mathfrak{e}\| < \frac{\epsilon}{Mu}$ for $u \in \mathbb{N}$. Since $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ is $\mathfrak{e}$-convergent sequence in $\mathcal{M}$. So there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\mathtt{d}^\mathtt{E}(\varkappa_\nu, p) \lll \frac{\mathfrak{e}}{2} \text{ for all } \nu \geq N_1$$

and

$$\mathtt{d}^\mathtt{E}(\varkappa_\nu, q) \lll \frac{\mathfrak{e}}{2} \text{ for all } \nu \geq N_2.$$

So,

$$\begin{aligned} \mathtt{d}^\mathtt{E}(p,q) &\preceq \mathtt{d}^\mathtt{E}(p, \varkappa_\nu) + \mathtt{d}^\mathtt{E}(\varkappa_\nu, q) \\ &\lll \frac{\mathfrak{e}}{2} + \frac{\mathfrak{e}}{2} = \mathfrak{e}. \end{aligned}$$

Since $0_\mathtt{E} \lll \mathtt{d}^\mathtt{E}(p,q) \lll \mathfrak{e}$, by [8, Proposition 1], $0_\mathtt{E} \preceq \mathtt{d}^\mathtt{E}(p,q) \preceq \mathfrak{e}$ and $\mathtt{E}^+$ is a normal cone, we have

$$\left\|\mathtt{d}^\mathtt{E}(p,q)\right\| \leq M\|\mathfrak{e}\| < \frac{\epsilon}{u}.$$

Applying limit $u \to +\infty$ it follows $\frac{\epsilon}{u} \to 0$, so

$$\mathtt{d}^\mathtt{E}(p,q) = 0 \text{ implies } p = q,$$

which contradicts our hypothesis. □

Proposition 3.5. *Let* $(\mathcal{M}, \mathtt{d}^\mathtt{E})$ *be an* $\mathtt{E}$*-metric space and* $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ *is* $\mathfrak{e}$*-convergent to* $\varkappa \in \mathcal{M}$ *then* $\mathtt{E}^+$ *is not a normal cone provided that* $\mathtt{d}^\mathtt{E}(\varkappa_\nu, \varkappa) \nrightarrow 0_\mathtt{E}$ $(\nu \to +\infty)$.

Proof. Assume that $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa) \nrightarrow 0_{\mathtt{E}}$ $(\nu \to +\infty)$. Suppose on contrary that $\mathtt{E}^+$ is a normal cone with normal constant M. Start with $\epsilon > 0$ and let $\mathfrak{e} \in (\mathtt{E}^+)^{\ominus}$ with $\|\mathfrak{e}\| < \frac{\epsilon}{M}$. Since $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ is an $\mathfrak{e}$-convergent sequence in $\mathcal{M}$, so there exists $N_1 \in \mathbb{N}$ such that

$$\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa) \lll \mathfrak{e}, \text{ for all } \nu \geq N_1,$$

and

$$\left\|\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa)\right\| \leq M\|\mathfrak{e}\| < \varepsilon.$$

This means $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa) \to 0_{\mathtt{E}}$ $(\nu \to +\infty)$, which is a contradiction to our hypothesis. □

Proposition 3.6. *Let $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ be an $\mathtt{E}$-metric space and $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ be an $\mathfrak{e}$-Cauchy sequence in $\mathcal{M}$ then $\mathtt{E}^+$ is not a normal cone provided that*

$$\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu) \nrightarrow 0_{\mathtt{E}} \ \text{as} \ (\nu, \mu \to +\infty).$$

Proof. Assume that $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu) \nrightarrow 0_{\mathtt{E}}$ $(\nu, \mu \to +\infty)$. Suppose on contrary that $\mathtt{E}^+$ is a normal cone having normal constant M. Start with $\epsilon > 0$ and let $\mathfrak{e} \in (\mathtt{E}^+)^{\ominus}$ with $\|\mathfrak{e}\| < \frac{\epsilon}{M}$. Since $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ is an $\mathfrak{e}$-Cauchy sequence in $\mathcal{M}$, so there exists $N_1 \in \mathbb{N}$ such that

$$\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu) \lll \mathfrak{e}, \text{ for all } \nu, \mu \geq N_1,$$

and

$$\left\|\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu)\right\| \leq M\|\mathfrak{e}\| < \varepsilon.$$

This means $\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_\mu) \to 0_{\mathtt{E}}$, which is a contradiction to our hypothesis. □

4. The E-distance space

In this section, we are going to define E-distance space that is the mixed form of both concepts defined in [3] and [5].

Definition 4.1. [5] Let $\mathcal{M}$ be a non-empty set and $\mathtt{d} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$, then $\mathtt{d}$ is a distance on $\mathcal{M}$, if

(1) $\mathtt{d}(\varkappa, \eta) \geq 0$ for all $\varkappa, \eta \in \mathcal{M}$,
(2) If $\mathtt{d}(\varkappa, \eta) + \mathtt{d}(\eta, \varkappa) = 0$ then $\varkappa = \eta$ for all $\varkappa, \eta \in \mathcal{M}$.
(3) If $\varkappa = \eta$ then $\mathtt{d}(\varkappa, \eta) = 0$, for all $\varkappa, \eta \in \mathcal{M}$.

Definition 4.2. [5] Let $\mathcal{M}$ be a non-empty set. A function $\mathtt{d} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is called s-distance on $\mathcal{M}$ if $\mathtt{d}$ satisfies the following axioms:

(1) $\mathtt{d}(\varkappa, \eta) \geq 0$, for all $\varkappa, \eta \in \mathcal{M}$.
(2) $\mathtt{d}(\varkappa, \eta) + \mathtt{d}(\eta, \varkappa) = 0$ if and only if $\varkappa = \eta$, for all $\varkappa, \eta \in \mathcal{M}$.
(3) For a positive real number s,

$$\mathtt{d}(\varkappa, \eta) \leq s[\mathtt{d}(\varkappa, \vartheta) + \mathtt{d}(\vartheta, \eta)], \text{ for all } \varkappa, \eta, \vartheta \in \mathcal{M}.$$

Then $(\mathcal{M}, \mathtt{d})$ is called a symmetric s-distance space and $\mathtt{d}$ is called a symmetric s-distance on $\mathcal{M}$ if $\mathtt{d}(\varkappa, \eta) = \mathtt{d}(\eta, \varkappa)$, for all $\varkappa, \eta \in \mathcal{M}$.

Remark 4.1. (1) *In s-distance space, $\mathtt{d}$ is not necessarily a continuous function.*

(2) *Limit of convergent sequence may not be unique in an s-distance space.*

Now, we define $\mathtt{E}$-distance space.

Definition 4.3. Let $\mathcal{M}$ be a nonempty set and $\mathtt{E}$ an ordered space. An ordered $\mathtt{E}$-distance on $\mathcal{M}$ is a function $\mathtt{d}^{\mathtt{E}} : \mathcal{M} \times \mathcal{M} \to \mathtt{E}$ such that for all $\varkappa, \eta \in \mathcal{M}$, we have

(1) $0_{\mathtt{E}} \preceq \mathtt{d}^{\mathtt{E}}(\varkappa, \eta)$,
(2) $\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) + \mathtt{d}^{\mathtt{E}}(\eta, \varkappa) = 0_{\mathtt{E}}$ if and only if $\varkappa = \eta$.

Then the pair $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ is called $\mathtt{E}$-distance space.

Now, we give an example of $\mathtt{E}$-distance space.

Example 4.4. Let $\mathcal{M} = \mathbb{N}$, $\mathtt{E} = \mathbb{R}^2$ with the ordering

$$(\varkappa_1, \varkappa_2) \leq (\eta_1, \eta_2) \text{ if and only if } \varkappa_i \leq \eta_i \text{ for } i = 1, 2.$$

For all $\mu > \nu$, we define

$$\begin{aligned} \mathtt{d}^{\mathtt{E}}(\mu, \nu) &= (\mu - \nu, \mu - \nu), \\ \mathtt{d}^{\mathtt{E}}(\nu, \mu) &= \left(\nu^{-1} - \mu^{-1}, \nu^{-1} - \mu^{-1}\right). \end{aligned}$$

Since

$$0_{\mathtt{E}} = (0, 0) \leq (\mu - \nu, \mu - \nu) \text{ and } 0_{\mathtt{E}} = (0, 0) \leq \left(\nu^{-1} - \mu^{-1}, \nu^{-1} - \mu^{-1}\right),$$

we have

$$0_{\mathtt{E}} \leq \mathtt{d}^{\mathtt{E}}(\mu, \nu) \text{ and also } 0_{\mathtt{E}} \leq \mathtt{d}^{\mathtt{E}}(\nu, \mu).$$

Consider

$$\begin{aligned} (0, 0) &= \mathtt{d}^{\mathtt{E}}(\mu, \nu) + \mathtt{d}^{\mathtt{E}}(\nu, \mu) \\ &= (\mu - \nu, \mu - \nu) + \left(\nu^{-1} - \mu^{-1}, \nu^{-1} - \mu^{-1}\right) \\ &= \left(\mu - \nu + \nu^{-1} - \mu^{-1}, \mu - \nu + \nu^{-1} - \mu^{-1}\right), \end{aligned}$$

that is

$$\mu - \nu + \nu^{-1} - \mu^{-1} = 0 \text{ and } \mu - \nu + \nu^{-1} - \mu^{-1} = 0$$

which implies $\mu = \nu$. Both axioms of $\mathtt{E}$-distance space are satisfied. Hence $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ is an $\mathtt{E}$-distance space.

Definition 4.5. Let $\mathtt{d}^{\mathtt{E}}$ be an $\mathtt{E}$-distance on $\mathcal{M}$ and $s \geq 1$, then for $\mathtt{d}^{\mathtt{E}}$ we say that is the $\mathtt{E}(\mathtt{s})$-distance on $\mathcal{M}$ if holds

$$\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) \preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa, \vartheta) + \mathtt{d}^{\mathtt{E}}(\vartheta, \eta)\right], \tag{1}$$

for all $\varkappa, \eta, \vartheta \in \mathcal{M}$.

Now, we give an example of symmetric $\mathtt{E}(\mathtt{s})$-distance space.

Example 4.6. Let $\mathcal{M} = \{1, 2, 3, 4\}$ and $\mathtt{E} = \{(0, \varkappa) : \varkappa \in \mathbb{R}\}$ be a subspace of $\mathbb{R}^2$. Then $\mathtt{E}^+ = \{(0, \varkappa) : \varkappa \geq 0\}$. Define $\mathtt{d}^{\mathtt{E}} : \mathcal{M} \times \mathcal{M} \to \mathtt{E}$ by

$$\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) = \begin{cases} \left(0, \frac{1}{|\varkappa - \eta|}\right) & \varkappa \neq \eta \\ (0, 0) & \varkappa = \eta, \end{cases}$$

then $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ is symmetric $\mathtt{E}(\mathtt{s})$-distance space with $s = \frac{6}{5}$.

5. The coincidence point results for $\mathtt{E}(\mathtt{s})$-distance space

In this section, we intend to prove the existence of a coincidence point of $\digamma$ and G, with $\mathcal{M}$ being an $\mathtt{E}(\mathtt{s})$-distance space.

Theorem 5.1. *Let $\digamma$ and G be two self mappings on a symmetric $\mathtt{E}(\mathtt{s})$-distance space $\mathcal{M}$ with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \emptyset$, satisfying*

$$\mathtt{d}^{\mathtt{E}}(\digamma\varkappa, \digamma\eta) \preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa, G\eta) + l\mathtt{d}^{\mathtt{E}}(G\eta, \digamma\varkappa),$$

for all $\varkappa, \eta \in \mathcal{M}$, where k and l are non negative reals with $k + ls < 1$. If range of $\digamma$ is contained in range of G and $G(\mathcal{M})$ is $\mathfrak{c}$-complete subspace of $\mathcal{M}$ then $\digamma$ and G have a coincidence point in $\mathcal{M}$.

Proof. Let $\varkappa_0$ be an arbitrary point in $\mathcal{M}$, since $\digamma(\mathcal{M}) \subset G(\mathcal{M})$ choose $\varkappa_1 \in \mathcal{M}$ such that $\digamma\varkappa_0 = G\varkappa_1$. So, we have, for $\varkappa_\nu \in \mathcal{M}$, we have $\varkappa_{\nu+1} \in \mathcal{M}$ such that

$$G\varkappa_{\nu+1} = \digamma\varkappa_\nu.$$

Now, we obtain

$$\begin{aligned} \mathtt{d}^{\mathtt{E}}(G\varkappa_\nu, G\varkappa_{\nu+1}) &= \mathtt{d}^{\mathtt{E}}(\digamma\varkappa_{\nu-1}, \digamma\varkappa_\nu) \\ &\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) + l\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu, \digamma\varkappa_\nu) \\ &\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) + l\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu, G\varkappa_{\nu+1}) \\ &\vdots \\ &\preceq \left(\frac{k}{1-l}\right)^\nu \mathtt{d}^{\mathtt{E}}(G\varkappa_0, G\varkappa_1). \end{aligned}$$

Since $l+k<1$, we have $\frac{k}{1-l}<1$. Let $\beta=\frac{k}{1-l}$, for $\nu>\mu$, we have

$$\begin{aligned}\mathtt{d}^{\mathtt{E}}(G\varkappa_\mu,G\varkappa_\nu) &\preceq s\left[\mathtt{d}^{\mathtt{E}}(G\varkappa_\mu,G\varkappa_{\mu+1})+\mathtt{d}^{\mathtt{E}}(G\varkappa_{\mu+1},G\varkappa_\nu)\right]\\ &\preceq s\mathtt{d}^{\mathtt{E}}(G\varkappa_\mu,G\varkappa_{\mu+1})+s^2\mathtt{d}^{\mathtt{E}}(G\varkappa_{\mu+1},G\varkappa_{\mu+2})+\cdots\\ &\quad +s^{\nu-\mu}\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1},G\varkappa_\nu)\\ &\preceq \left[s\beta^\mu+s^2\beta^{\mu+1}+\cdots+s^{\nu-\mu}\beta^{\nu-1}\right]\mathtt{d}^{\mathtt{E}}(G\varkappa_0,G\varkappa_1)\\ &\preceq s\beta^\mu\frac{1-(s\beta)^{\nu-\mu}}{1-s\beta}\mathtt{d}^{\mathtt{E}}(G\varkappa_0,G\varkappa_1).\end{aligned}$$

Given $\mathfrak{e}\ggg 0_{\mathtt{E}}$, choose $\rho>0$ such that $\mathfrak{e}-\rho U_+\subseteq \mathtt{E}^+$ and $k_1\in\mathbb{N}$ such that

$$s\beta^\mu\frac{1-(s\beta)^{\nu-\mu}}{1-s\beta}\mathtt{d}^{\mathtt{E}}(G\varkappa_0,G\varkappa_1)\in\frac{\rho}{2}U_+,$$

for any $\mu,\nu\geq k_1$, thus we have,

$$\mathfrak{e}-s\beta^\mu\frac{1-(s\beta)^{\nu-\mu}}{1-s\beta}\mathtt{d}^{\mathtt{E}}(G\varkappa_0,G\varkappa_1)-\frac{\rho}{2}U_+\subseteq\mathfrak{e}-\rho U_+\subseteq\mathtt{E}^+,$$

therefore

$$\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu,G\varkappa_\mu)\preceq s\beta^\mu\frac{1-(s\beta)^{\nu-\mu}}{1-s\beta}\mathtt{d}^{\mathtt{E}}(G\varkappa_0,G\varkappa_1)\lll\mathfrak{e},$$

for all $\nu,\mu\geq k_1$, which implies the sequence $(G\varkappa_\nu)_{\nu\in\mathbb{N}}$ is $\mathfrak{e}$-Cauchy. Because $G\mathcal{M}$ is $\mathfrak{e}$-complete so there will some $\varkappa\in\mathcal{M}$ such that $G\varkappa_\nu\xrightarrow{\mathfrak{e}}G\varkappa$. Now for $\mathfrak{e}\ggg 0_{\mathtt{E}}$, choose k_2, such that $\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)\lll\frac{\mathfrak{e}}{k+1}$ for all $\nu\geq k_2$. Consider for all $\nu\geq k_2$,

$$\begin{aligned}\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu,F\varkappa) &= \mathtt{d}^{\mathtt{E}}(F\varkappa_{\nu-1},F\varkappa)\\ &\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1},G\varkappa)+l\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa_{\nu-1})\\ &\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1},G\varkappa)+l\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)\\ &\lll k\frac{\mathfrak{e}}{k+l}+l\frac{\mathfrak{e}}{k+l}\\ &\lll \mathfrak{e}.\end{aligned}$$

Which implies $\lim_{\nu\to\infty}G\varkappa_\nu=F\varkappa$. Now, we have

$$\begin{aligned}\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa) &\preceq s\left[\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)+\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu,F\varkappa)\right]\\ &\preceq s\left[\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)+\mathtt{d}^{\mathtt{E}}(F\varkappa_{\nu-1},F\varkappa)\right]\\ &\preceq s\left[\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)+k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1},G\varkappa)+l\mathtt{d}^{\mathtt{E}}(F\varkappa_{\nu-1},G\varkappa)\right]\\ &\preceq s\left[(1+l)\mathtt{d}^{\mathtt{E}}(G\varkappa,G\varkappa_\nu)+k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1},G\varkappa)\right]\\ &\lll \mathfrak{e},\end{aligned}$$

for all $\nu\geq k_2$. Since $\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa)\lll\frac{\mathfrak{e}}{\mu}$for any $\mu\geq 1$, for any $\frac{\mathfrak{e}}{\mu}\ggg 0_{\mathtt{E}}$, $\frac{\mathfrak{e}}{\mu}-\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa)\in\mathtt{E}^+$ for all $\mu\in\mathbb{N}$ which implies $-\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa)\in\mathtt{E}^+$ but $\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa)\in\mathtt{E}^+$, thus we have $\mathtt{d}^{\mathtt{E}}(G\varkappa,F\varkappa)=0_{\mathtt{E}}$. Hence $\varkappa$ is coincidence point of G and F. □

Corollary 5.2. *Let $\digamma$ and G be two self mappings on a symmetric* $\mathtt{E(s)}$*-distance space $\mathcal{M}$ with closed positive cone $\mathtt{E}^+$ and $(\mathtt{E}^+)^{\ominus} \neq \emptyset$, such that*

$$\mathtt{d}^{\mathtt{E}}(\digamma\varkappa, \digamma\eta) \preceq p\left[\mathtt{d}^{\mathtt{E}}(G\varkappa, G\eta) + \mathtt{d}^{\mathtt{E}}(G\eta, \digamma\varkappa)\right]$$

for all $\varkappa, \eta \in \mathcal{M}$, where $p \in \left[0, \frac{1}{1+s}\right)$. If range of $\digamma$ is contained in range of G and $G(\mathcal{M})$ is $\mathfrak{c}$-complete subspace of $\mathcal{M}$, then $\digamma$ and G have at least a coincidence point in $\mathcal{M}$.

In 1998, Jungck and Rhoades [12] introduced the notion of weakly compatible maps. The compatible maps (see [11]) are weakly compatible but converse need not be true (see also [6]).

Definition 5.1. [12] A pair of maps $\digamma$ and G is called weakly compatible pair if they commute at coincidence points.

Using the concept of weakly compatible maps and Theorem 5.1 we get the following result.

Theorem 5.3. *Let $\digamma$ and G be two self mappings on a symmetric* $\mathtt{E(s)}$*-distance space $\mathcal{M}$ with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \emptyset$, satisfying*

$$\mathtt{d}^{\mathtt{E}}(\digamma\varkappa, \digamma\eta) \preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa, G\eta) + l\mathtt{d}^{\mathtt{E}}(G\varkappa, \digamma\varkappa)$$

for all $\varkappa, \eta \in \mathcal{M}$, where $k, l \geq 0$ with $k + ls < 1$. If range of $\digamma$ is contained in range of G and $G(\mathcal{M})$ is $\mathfrak{c}$-complete subspace of $\mathcal{M}$. Then $\digamma$ and G have a coincidence point in $\mathcal{M}$. Moreover, if $\digamma$ and G are weakly compatible, then $\digamma$ and G have a unique common fixed point in $\mathcal{M}$.

PROOF. Take $\varkappa_0$, an arbitrary point of $\mathcal{M}$, since $\digamma(\mathcal{M}) \subset G(\mathcal{M})$ choose $\varkappa_1 \in \mathcal{M}$ such that $\digamma\varkappa_0 = G\varkappa_1$. Continuing in this way, for $\varkappa_\nu \in \mathcal{M}$, we have $\varkappa_{\nu+1} \in \mathcal{M}$ such that

$$G\varkappa_{\nu+1} = \digamma\varkappa_\nu.$$

Consider

$$\begin{aligned}
\mathtt{d}^{\mathtt{E}}(G\varkappa_\nu, G\varkappa_{\nu+1}) &= \mathtt{d}^{\mathtt{E}}(\digamma\varkappa_{\nu-1}, \digamma\varkappa_\nu) \\
&\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) + l\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, \digamma\varkappa_{\nu-1}) \\
&\preceq k\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) + l\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) \\
&\preceq (k+l)\,\mathtt{d}^{\mathtt{E}}(G\varkappa_{\nu-1}, G\varkappa_\nu) \\
&\quad \vdots \\
&\preceq (k+l)^\nu\,\mathtt{d}^{\mathtt{E}}(G\varkappa_0, G\varkappa_1).
\end{aligned}$$

In similar manner as of the above Theorem, we have at least a point of coincidence of G and $\digamma$, say $\omega = Gp = \digamma p$, $p \in \mathcal{M}$. Now for uniqueness assume there exists

$q \in \mathcal{M}$ such that $\digamma q = Gq$, then

$$\begin{aligned} \mathtt{d}^{\mathtt{E}}(Gp, Gq) &= \mathtt{d}^{\mathtt{E}}(\digamma p, \digamma q) \\ &\preceq k\mathtt{d}^{\mathtt{E}}(Gp, Gq) + l\mathtt{d}^{\mathtt{E}}(Gp, \digamma p) \\ &\preceq k\mathtt{d}^{\mathtt{E}}(Gp, Gq) + l\mathtt{d}^{\mathtt{E}}(Gp, Gp). \end{aligned}$$

Thus we have,

$$(1-k)\,\mathtt{d}^{\mathtt{E}}(Gp, Gq) \preceq 0.$$

So, $-\mathtt{d}^{\mathtt{E}}(Gp, Gq) \in \mathtt{E}^{+}$, since $\mathtt{d}^{\mathtt{E}}(Gp, Gq) \in \mathtt{E}^{+}$ we conclude,

$$\mathtt{d}^{\mathtt{E}}(Gp, Gq) = 0.$$

So, ω is a unique point of coincidence of G and $\digamma$. Since G and $\digamma$ are weakly compatible self mappings, ω is a unique common fixed point of G and $\digamma$. □

Corollary 5.4. *Let $\digamma$ and G two self mappings on a symmetric $\mathtt{E(s)}$-distance space $\mathcal{M}$ with closed positive cone $\mathtt{E}^{+}$ so that $(\mathtt{E}^{+})^{\ominus} \neq \emptyset$ satisfying*

$$\mathtt{d}^{\mathtt{E}}(\digamma\varkappa, \digamma\eta) \preceq b\left[\mathtt{d}^{\mathtt{E}}(G\varkappa, G\eta) + \mathtt{d}^{\mathtt{E}}(G\varkappa, \digamma\varkappa)\right],$$

for all $\varkappa, \eta \in \mathcal{M}$, where $b \in \left[0, \frac{1}{1+s}\right)$. If range of $\digamma$ is contained in range of G and $G(\mathcal{M})$ is $\mathfrak{c}$-complete subspace of $\mathcal{M}$. Then $\digamma$ and G has at least a coincidence point in $\mathcal{M}$. Moreover, if $\digamma$ and G are weakly compatible, then $\digamma$ and G have a unique common fixed point in $\mathcal{M}$. In either case, for any $\varkappa_0 \in \mathcal{M}$, $\{G\varkappa_\nu\}$ converges to unique common fixed point of G and $\digamma$.

Remark 5.5. *Note that Theorem 5.1 represents a generalization of Banach's fixed point theorem and also generalizes some results of Jungck (see [**10, 11**]).*

Remark 5.6. *All the above fixed point results remains valid if we consider $\mathtt{E(s)}$-distance space instead of symmetric $\mathtt{E(s)}$-distance space. However for that case positive cone should be normal one.*

6. The rational type contraction in $\mathtt{E(s)}$-distance space

In this section, we intend to prove existence of fixed point result in an $\mathtt{E(s)}$-distance space in the presence of rational type contractive condition by using comparison function ψ.

Theorem 6.1. *Let $(\mathcal{M}, \mathtt{d}^{\mathtt{E}})$ be a $\mathfrak{c}$-complete symmetric $\mathtt{E(s)}$-distance space with closed cone $\mathtt{E}^{+}$ such that $(\mathtt{E}^{+})^{\ominus} \neq \emptyset$ and a self mapping $\digamma$ on $\mathcal{M}$ satisfying*

$$\mathtt{d}^{\mathtt{E}}(\digamma\varkappa, \digamma\eta) \preceq \lambda\mathtt{d}^{\mathtt{E}}(\varkappa, \eta) + \frac{\mu\left[1 + \psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa, \digamma\varkappa)\right)\right]\mathtt{d}^{\mathtt{E}}(\eta, \digamma\eta)}{1 + \psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa, \eta)\right)}, \tag{2}$$

for all $\varkappa, \eta \in \mathcal{M}$, where λ and μ are non negative reals with $\lambda + s\mu < 1$ a and $\psi : \mathtt{E}^{+} \to \mathbb{R}$ is continuous, $\psi(0_{\mathtt{E}}) = 0$. Then $\digamma$ has a fixed point in $\mathcal{M}$ provided that positive cone is normal.

PROOF. Let $\varkappa_0$ be an arbitrary point in $\mathcal{M}$, consider the iterative sequence

$$\varkappa_1 = \digamma\varkappa_0, \varkappa_2 = \digamma\varkappa_1, \ldots, \varkappa_{\nu+1} = \digamma\varkappa_\nu,$$

then

$$\begin{aligned} \mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_{\nu+1}) &= \mathtt{d}^{\mathtt{E}}(\digamma\varkappa_{\nu-1}, \digamma\varkappa_\nu) \\ &\preceq \lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu) + \frac{\mu\left[1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \digamma\varkappa_{\nu-1})\right)\right]\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \digamma\varkappa_\nu)}{1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu)\right)} \\ &= \lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu) + \frac{\mu\left[1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu)\right)\right]\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_{\nu+1})}{1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu)\right)} \\ &= \lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu) + \mu\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_{\nu+1}) \\ &\preceq \frac{\lambda}{1-\mu}\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu). \end{aligned}$$

Now, with $\gamma = \frac{\lambda}{1-\mu} < 1$, we have

$$\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_{\nu+1}) \preceq \gamma\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu) \preceq \ldots \preceq \gamma^\nu\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1).$$

So, for any $\nu > \mu$, we have

$$\begin{aligned} \mathtt{d}^{\mathtt{E}}(\varkappa_\mu, \varkappa_\nu) &\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa_\mu, \varkappa_{\mu+1}) + \mathtt{d}^{\mathtt{E}}(\varkappa_{\mu+1}, \varkappa_\nu)\right] \\ &\preceq s\mathtt{d}^{\mathtt{E}}(\varkappa_\mu, \varkappa_{\mu+1}) + s^2\mathtt{d}^{\mathtt{E}}(\varkappa_{\mu+1}, \varkappa_{\mu+2}) + \cdots + s^{\nu-\mu}\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1}, \varkappa_\nu) \\ &\preceq \left[s\gamma^\mu + s^2\gamma^{\mu+1} + \cdots + s^{\nu-\mu}\gamma^{\nu-1}\right]\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1) \\ &\preceq s\gamma^\mu\left[1 + s\gamma + s^2\gamma^2 + \cdots + (s\gamma)^{\nu-\mu-1}\right]\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1) \\ &\preceq s\gamma^\mu\frac{1-(s\gamma)^{\nu-\mu}}{1-s\gamma}\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1). \end{aligned}$$

For $\mathfrak{e} \ggg 0_{\mathtt{E}}$, choose $\rho > 0$ so that $\mathfrak{e} - \rho U_+ \subseteq \mathtt{E}^+$ and $k_1 \in \mathbb{N}$ such that

$$s\gamma^\mu\frac{1-(s\gamma)^{\nu-\mu}}{1-s\gamma}\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1) \in \frac{\rho}{2}U_+$$

for any $\mu, \nu \geq k_1$, thus $\mathfrak{e} - s\gamma^\mu\frac{1-(s\gamma)^{\nu-\mu}}{1-s\gamma}\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1) - \frac{\rho}{2}U_+ \subseteq \mathfrak{e} - \rho U_+ \subseteq \mathtt{E}^+$, therefore

$$\mathtt{d}^{\mathtt{E}}(\varkappa_\mu, \varkappa_\nu) \preceq s\gamma^\mu\frac{1-(s\gamma)^{\nu-\mu}}{1-s\gamma}\mathtt{d}^{\mathtt{E}}(\varkappa_0, \varkappa_1) \lll \mathfrak{e} \text{ for all } \nu, \mu \geq k_1.$$

Which implies $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ is a $\mathfrak{e}$-Cauchy sequence, since $\mathcal{M}$ is $\mathfrak{e}$-complete so there exists some $\varkappa \in \mathcal{M}$ such that $\varkappa_\nu \xrightarrow{\mathfrak{e}} \varkappa$. For a given $\mathfrak{e} \ggg 0_{\mathtt{E}}$, choose a natural k_2, such that $\mathtt{d}^{\mathtt{E}}(\varkappa, \varkappa_\nu) \lll \frac{\mathfrak{e}(1-\mu)}{2s}$, $s\mathtt{d}^{\mathtt{E}}(\varkappa_\nu, \varkappa_{\nu+1}) \lll \frac{\mathfrak{e}(1-\mu)}{2s}$ for all $\nu \geq k_2$. Fixed point of this mapping $\digamma$ exists when we have a normal cone. Because in that case if the sequence $(\varkappa_\nu)_{\nu\in\mathbb{N}}$ is $\mathfrak{e}$-converges to $\varkappa$ then $\lim_{\nu\to\infty} \mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)=0$. Consider

$$\begin{aligned}
\mathtt{d}^{\mathtt{E}}(\varkappa,F\varkappa) &\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)+\mathtt{d}^{\mathtt{E}}(\varkappa_\nu,F\varkappa)\right]\\
&\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)+\mathtt{d}^{\mathtt{E}}(F\varkappa_{\nu-1},F\varkappa)\right]\\
&\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)+\lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)+\right.\\
&\qquad\left.\frac{\mu\left[1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},F\varkappa_{\nu-1})\right)\right]\mathtt{d}^{\mathtt{E}}(\varkappa,F\varkappa)}{1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)\right)}\right]\\
&\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)+\lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)+\right.\\
&\qquad\left.\frac{\mu\left[1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa_\nu)\right)\right]\mathtt{d}^{\mathtt{E}}(\varkappa,F\varkappa)}{1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)\right)}\right]\\
&\preceq s\left[\mathtt{d}^{\mathtt{E}}(\varkappa,\varkappa_\nu)+\lambda\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)+\right.\\
&\qquad\left.\frac{\mu\left[1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa_\nu)\right)\right]\mathtt{d}^{\mathtt{E}}(\varkappa,F\varkappa)}{1+\psi\left(\mathtt{d}^{\mathtt{E}}(\varkappa_{\nu-1},\varkappa)\right)}\right],\\
&\preceq s\mu\mathtt{d}^{\mathtt{E}}(\varkappa,F\varkappa).
\end{aligned}$$

So, $\varkappa$ is a fixed point of F. □

Remark 6.2. *Note that Theorem 6.1 generalized in several directions Theorem 1 in* [7].

7. Conclusion

In this article we define $\mathtt{E}(\mathtt{s})$-distance space and generalize the results of [Mehmood, N. et al, Positivity, (2019). 1-11.] in the settings of $\mathtt{E}(\mathtt{s})$-distance space with non-solid and non-normal cones. Also, some coincidence point results have been obtained that extend and generalize some known results in the literature. We also expect applications of these spaces in fixed point theoru, approximation theory, variational problems, optimization theory and so on.

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