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Fixed point results for generalized contractions in S-metric spaces

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Abstract. In this paper, we discuss the existence of a fixed point for a generalized contraction in S-metric spaces. We furnish some examples in support of our main results. Our results generalize and improve many well-known results in the existing literature.

1. Introduction

Fixed point theory originated with the method of successive approximations. In 1837, Liouville [9] and Picard [6] used the method of successive approximation to prove the existence of solutions for differential equations. Brouwer introduced the concept of the fixed point in the nineteenth century and named it Brouwer's fixed point theorem cite 3. In 1906, French mathematician Maurice Fréchet introduced the concept of metric space axiomatically. In 1922, the Polish mathematician Banach [1] established an important metric fixed point result regarding a contraction mapping, known as the Banach contraction principle. This principle is considered as one of the most remarkable results in analysis. It confirms the existence and uniqueness of fixed point of certain self maps on metric spaces. Due to its importance and clarity, several authors have achieved many interesting extensions and generalizations of this contraction principle. There is a vast amount of literature dealing with extensions and generalizations of this fundamental principle. Many mathematicians studied the Banach contraction principle and used it in many mathematical rules and mathematical science, like approximation theory, game theory, quantum theory, economics, differential equations and integral equations.

One way to generalize the Banach contraction principle is to enlarge the class of metric spaces. Many researchers have shown keen interest in this regard and

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have introduced different ways to generalize metric spaces. For example, in 1963, Gahler [4] initiated the concept of 2-metric space. In 1992, Dhage [3] modified the concept of 2-metric space and introduced the concept of D-metric. In 2005, Mustafa and Sims [11] came up with a new structure of generalized metric spaces, which is called a G-metric space (Also, see [7] and [12]). Sedghi et al. [14] customized the concepts of D-metric space and initiated the concept of D^*-metric space and also proved a common fixed point theorem in D^*-metric space. In 2012, S. Sedghi et al. [13] introduced the concept of S-metric space, which is a generalization of G-metric space and D^*-metric space.

In this article, we present a new method to find the existence of a fixed point, using a generalized contraction in S-metric spaces and we impart examples to validate our main result. We also derive some fixed point theorems for $\alpha - \psi$-contractions in an S-metric space.

Definition 1.1. [8] Let $X \neq \emptyset$. A mapping $d : X \times X \to [0, \infty)$ is called a metric on X, if

(i) $d(x, y) \geq 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$.

The pair (X, d) is called a metric space. The nonnegative real number $d(x, y)$ associated with any two points 'x' and 'y' of X by d is called the distance between 'x' and 'y'.

Example 1.2. [8] Let $X = \mathbb{R}$ be the set of all real numbers and let $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by $d(x, y) = |x-y|$, where $|x-y|$ denotes the absolute value of the number $x - y$. Therefore, the pair (X, d) is a metric space.

Definition 1.3. [8] Let (X, d) be a complete metric space. A function $f : X \to X$ is called a contraction, if there exists $k < 1$ such that for any $x, y \in X$

$$kd(x, y) \geq d(f(x), f(y)).$$

Example 1.4. Consider the metric space (R, d) where d is the Euclidean distance, i.e., $d(x, y) = |x - y|$. The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{x}{a} + b$ is a contraction if $a > 1$. In this specific case, we can find a fixed point. Since a fixed means that $f(x) = x$, we want $x = \frac{x}{a} + b$. Solving for x gives $x = \frac{ab}{a-1}$.

Definition 1.5. [5] Let $X \neq \emptyset$. A mapping $d : X \times X \times X \to [0, \infty)$ is called a 2-metric on X, if for all $x, y, z, a \in X$ it satisfies the following conditions:

(i) For distinct points $x, y \in X$, there is a point $z \in X$ such that $d(x, y, z) \neq 0$;
(ii) $d(x, y, z) = 0$ if any two elements of the triplet $x, y, z \in X$ are equal;

(iii) $d(x, y, z) = d(x, z, y) = \cdots$ (symmetry in all three variables);
(iv) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z \in X$.

The pair (X, d) is called a 2-metric space.

Definition 1.6. [5] Let $X \neq \emptyset$. A mapping $D : X \times X \times X \to [0, \infty)$ is called a D-metric on X, if for all $x, y, z, a \in X$:

(i) $D(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry in all three variables);
(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

The pair (X, D) is called a D-metric space.

Example 1.7. Define the functions σ and ρ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, for any $n \in \mathbb{N}$ by:

$$\sigma(x, u, z) = k \min \{||x - y||, ||y - z||, \|z - x\|\}, k > 0$$

and

$$\rho(x, y, z) = c \{\|x - y\| + \|y - z\| + \|z - x\|\}, c > 0$$

for all $x, y, z \in \mathbb{R}^n$, where $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$. Then $(\mathbb{R}^n, \sigma)$ and $(\mathbb{R}^n, \rho)$ are D-metric spaces.

Definition 1.8. [11] Let $X \neq \emptyset$. A mapping $G : X \times X \times X \to [0, \infty)$ is called a G-metric on X, if for all $x, y, z, a \in X$:

(i) $G(x, y, z) = 0$ if $x = y = z$;
(ii) $0 < G(x, x, y)$; with $x \neq y$;
(iii) $G(x, x, y) \leq G(x, y, z)$, with $z \neq y$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, (rectangle inequality).

The pair (X, G) is called a G-metric space.

Example 1.9. [11] Let $X = \{a, b\}$ and $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$ and $G(a, b, b) = 2$. Extend G to all of $X \times X \times X$ by symmetry in the variables. Then it is easily verified that G is a G-metric, but $G(a, b, b) \neq G(a, a, b)$.

Definition 1.10. [11] Let $X \neq \emptyset$. A mapping $D^* : X \times X \times X \to [0, \infty)$ is called a generalized metric (or D^*-metric) on X, if for all $x, y, z, a \in X$:

(i) $D^*(x, y, z) \geq 0$;
(i) $D^*(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function.
(iii) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Example 1.11. Some example of such a function are

(a) $D^*(x,y,z) = \min\{d(x,y), d(y,z), d(z,x)\}$,
(b) $D^*(x,y,z) = d(x,y) + d(y,z) + d(z,x)$,

where d is the ordinary metric on X.

Remark 1.1. [13] *It is easy to see that every G-metric is a D^*-metric, but the converse does not hold in general.*

Example 1.12. [13] If $X = \mathbb{R}$, define

$$D^*(x,y,z) = |x+y-2z| + |x+z-2y| + |y+z-2x|.$$

It is easy to see that $(\mathbb{R}, D^*)$ is a D^*-metric, but it is not a G-metric. Set $x=5$, $y=-5$ and $z=0$, then $G(x,x,y) \leq G(x,y,z)$ does not hold.

Definition 1.13. [13] Let $X \neq \emptyset$. A mapping $S : X \times X \times X \to [0,\infty)$ is called an S-metric on X, if it for all $x,y,z,a \in X$.

(i) $S(x,y,z) \geq 0$;
(ii) $S(x,y,z) = 0$ if and only if $x=y=z$;
(iii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Example 1.14. Let $X \neq \emptyset$ and (X,d) be a metric space. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Example 1.15. Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X. Then $S(x,y,z) = \|z+y-2x\| + \|x-z\|$ is an S-metric on X.

Remark 1.2. [13] *It is easy to see that every D^*-metric is an S-metric, but in general the converse is not true.*

Definition 1.16. [13] Let (X,S) be an S-metric space and $A \subset X$.

(1) If, for every $x \in A$, there exists $r > 0$ such that $B_s(x,r) \subset A$, then the subset A is called an open subset of X.
(2) A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,y,z) < r$ for all $x,y \in A$.
(3) A sequence $\{x_n\}$ in X converges to X if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x,y,z) < \varepsilon$ for all $n \geq n_0$ and we denote this by $\lim\limits_{n\to\infty} = x$.
(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $m,n \geq n_0$.
(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.
(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_s(x,r) \subset A$. Then τ is a topology on X induced by the S-metric S.

Definition 1.17. [10] Let (X, S) be a S-metric space and $T : X \to X$ be a given mapping. We say that T is α-admissible if

$$\alpha(x, y, z) \geq 1 \implies \alpha(Tx, Ty, Tz) \geq 1,$$

for all $x, y, z \in X$.

Example 1.18. [10] Let $X = [0, \infty)$ and d be a metric on X. A mapping $S : X \times X \times X \to [0, \infty)$ defined by

$$S(x, y, z) = d(x, z) + d(y, z)$$

is an S-metric on X. Let $\alpha : X \times X \times X \to [0, \infty)$. Let T given by

$$Tx = \sqrt{x}$$

and for α defined as

$$\alpha(x, y, z) = e^{\max\{x,y\}-z} \quad if \quad \max\{x, y\} \geq z$$

and

$$\alpha(x, y, z) = 0 \quad if \quad \max\{x, y\} < z.$$

So, it is easy to see that T is α-admissible.

Definition 1.19. [2] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F_1) F is strictly increasing.

(F_2) For every sequence $\{x_n\} \subset \mathbb{R}^+$, we have $\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty$.

(F_3) There exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In what follows, $\mathfrak{F}$ stands for the family of all functions F which satisfies the above three conditions.

Definition 1.20. [2] Let (X, S) be an S-metric space. A mapping $T : X \to X$ is said to be an F-contraction if there is a number $\tau > 0$ and an $F \in \mathfrak{F}$ such that

$$S(Tx, Ty, Tz) > 0 \Rightarrow \tau + F(S(Tx, Ty, Tz)) \leq F(S(x, y, z)), \text{ for all } x, y, z \in X.$$

Theorem 1.3. [13] *Let (X, S) be a complete S-metric space and let*

$$B_s(x_0, r) = \{x \in X : S(x, x, x_0) < r\}, \text{ where } x_0 \in X \text{ and } r > 0.$$

Suppose that $F : B_s(x_0, r) \to X$ is a contraction with

$$S(F(x_0), F(x_0), x_0) < (1 - L)\frac{r}{2}.$$

Then F has a unique fixed point in $B_s(x_0, r)$.

Theorem 1.4. [13] *Let (X, S) be a complete S-metric space with $S(x, y, z) = \|x - y\| + \|y - z\|$ and let $\overline{B_s(r)}$ be the closed ball of radius $r > 0$, centered at zero in Banach space X with $F : \overline{B_s(r)} \to X$ a contraction and $F\partial(\overline{B_s(r)}) \subseteq \overline{B_s(r)}$. Then F has a unique fixed point in $\overline{B_s(r)}$.*

2. Main results

Definition 2.1. Let (X,S) be an S-metric space and let $\alpha : X\times X\times X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is called an F_α-contraction, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$S(Tx,Ty,Tz) > 0 \Longrightarrow \tau + F(\alpha(x,y,z)S(Tx,Ty,Tz)) \leq F(S(x,y,z)),$$

for all $x,y,z \in X$.

Theorem 2.1. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be an F_α-contraction, satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. Let $x_0 \in X$. Consider the sequence $\{x_n\}$ defined by

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, ...,\ x_n = Tx_{n-1} = T^nx_0.$$

By our assumption (ii), we know that $\alpha(x_0,x_0,Tx_0) \geq 1$ and as T is α-admissible, so,

$$\alpha(x_1,x_1,x_2) \geq 1$$

and by induction on n, we conclude that $\alpha(x_n,x_n,x_{n+1}) \geq 1$, for all n. Now,

$$\begin{aligned} S\left(x_n,x_{n,}x_{n+1}\right) &= S\left(Tx_{n-1},Tx_{n-1},Tx_n\right) \\ &\leq \alpha\left(x_{n-1},x_{n-1},x_n\right) S\left(Tx_{n-1},Tx_{n-1},Tx_n\right) \\ \Rightarrow F\left(S\left(x_n,x_n,x_{n+1}\right)\right) &\leq F\left(\alpha\left(x_{n-1},x_{n-1},x_n\right)\right) S\left(Tx_{n-1},Tx_{n-1},Tx_n\right)). \end{aligned}$$

So,

$$F\left(S\left(x_n,x_n,x_{n+1}\right)\right) \leq F\left(S\left(x_{n-1},x_{n-1},x_n\right)\right) - \tau. \tag{1}$$

Now,

$$\begin{aligned} S\left(x_{n-1},x_{n-1},x_n\right) &= S\left(Tx_{n-2},Tx_{n-2},Tx_{n-1}\right) \\ &\leq \alpha\left(x_{n-2},x_{n-2},x_{n-1}\right) S\left(Tx_{n-2},Tx_{n-2},Tx_{n-1}\right). \end{aligned}$$

Thus,

$$F(S\left(x_{n-1},x_{n-1},x_n\right)) \leq F(\alpha\left(x_{n-2},x_{n-2},x_{n-1}\right) S\left(Tx_{n-2},Tx_{n-2},Tx_{n-1}\right)).$$

Then

$$F(S\left(x_{n-1},x_{n-1},x_n\right)) \leq F(S\left(x_{n-2},x_{n-2},x_{n-1}\right) - \tau. \tag{2}$$

Putting (2) in (1), we have

$$F(S\left(x_n,x_n,x_{n+1}\right)) \leq F(S\left(x_{n-2},x_{n-2},x_{n-1}\right) - 2\tau.$$

Continuing in this way, we get,

$$F(S(x_n, x_n, x_{n+1})) \leq F(S(x_0, x_0, x_1) - n\tau. \quad (3)$$

Now, let

$$\partial_n = S(x_n, x_n, x_{n+1}).$$

So,

$$F(\partial_n) \leq F(\partial_0) - n\tau. \quad (4)$$

Now,

$$\begin{aligned} \lim_{n\to\infty} F(\partial_n) &\leq \lim_{n\to\infty} (F(\partial_0) - n\tau) \\ &\Rightarrow \lim_{n\to\infty} F(\partial_n) \leq -\infty \\ \Rightarrow \lim_{n\to\infty} (\partial_n) &= 0. \end{aligned}$$

So,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \quad (5)$$

Now, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\partial_n)^k F(\partial_n) = 0. \quad (6)$$

Now, using (4) implies that

$$\partial_n^k F(\partial_n) \leq \partial_n^k (F(\partial_0) - n\tau).$$

Adding and subtracting $\partial_n^k F(\partial_0)$ on left side of the above inequality, we get

$$\begin{aligned} \partial_n^k F(\partial_n) - \partial_n^k F(\partial_0) + \partial_n^k F(\partial_0) &\leq \partial_n^k (F(\partial_0) - n\tau) \\ \Rightarrow \partial_n^k F(\partial_n) - \partial_n^k F(\partial_0) &\leq -\partial_n^k n\tau. \end{aligned}$$

So,

$$\partial_n^k F(\partial_n) \leq \partial_n^k F(\partial_0) - \partial_n^k n\tau. \quad (7)$$

Now, by applying limit on both sides, we have

$$\begin{aligned} \lim_{n\to\infty} \partial_n^k F(\partial_n) &\leq \lim_{n\to\infty} \partial_n^k F(\partial_0) - \lim_{n\to\infty} \partial_n^k n\tau. \\ \Rightarrow 0 &= 0 - \lim_{n\to\infty} \partial_n^k n\tau. \\ \Rightarrow \lim_{n\to\infty} \partial_n^k n\tau &= 0. \\ \Rightarrow \partial_n^k n\tau &\leq 1. \end{aligned}$$

For $\tau = 1$,

$$\partial_n^k \leq \frac{1}{n}, \Rightarrow \left[\partial_n^k\right]^{\frac{1}{k}} \leq (\frac{1}{n})^{\frac{1}{k}} \Rightarrow \partial_n \leq \frac{1}{n^{\frac{1}{k}}}.$$

Now, consider $m, n \in \mathbb{N}$ and $m, n \geq n_0$, for some n_0. Then

$$\begin{aligned} S\left(x_m, x_m, x_n\right) &\leq \partial_{m-1} + \partial_{m-2} + ... + \partial_n \\ &\leq \sum_{i=n}^{\infty} \partial_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \leq \infty. \end{aligned}$$

Since the series is convergent, the sequence $\{x_n\}$ is convergent, i.e., $\lim_{n\to\infty} x_n = x^*$. Since T is continuous, we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^* \Rightarrow Tx^* = x^*.$$

So, T has a fixed point. □

Example 2.2. Let $X = \mathbb{R}$ and $S\left(x, y, z\right) = \mid x - z \mid + \mid y - z \mid$ be an S-metric on X. Suppose that

$$\alpha(x, y, z) = \begin{cases} e^{\max\{x,y\}-z}, & \text{if } \max\{x, y\} \geq z, \\ 0, & \text{if } \max\{x, y\} < z, \end{cases}$$

and

$$F(x) = \frac{1}{2} \sinh x, \; T(x, y, z) = 0.1.$$

We have to show that

$$\tau + F\left(\alpha\left(x, y, z\right) S\left(Tx, Ty, Tz\right)\right) \leq F(S\left(x, y, z\right)). \tag{8}$$

Take the right side of (8). Let $x = 0.3$, $y = 0.2$ and $z = 0.1$. We have

$$S\left(0.3, 0.2, 0.1\right) = \mid 0.3 - 0.1 \mid + \mid 0.2 - 0.1 \mid = 0.3.$$

Now,

$$F\left(0.3\right) = \frac{1}{2} \sinh\left(0.3\right) \Rightarrow F\left(0.3\right) = 0.30452029344.$$

Also,

$$\alpha\left(0.3, 0.2, 0.1\right) = e^{\max\{0.3,0.2\}-0.1} = e^{0.2} = 1.22.$$

Now, we have $T\left(0.3\right) = 0.1$, $T\left(0.2\right) = 0.1$ and $T\left(0.1\right) = 0.1$. So,

$$S\left(0.1, 0.1, 0.1\right) = 0.$$

Then,

$$F\left(\left(1.22\right)\left(0\right)\right) = F\left(0\right) = \frac{1}{2} \sinh\left(0\right) = 0.$$

Putting the values in (8),

$$0.001 + 0 \leq 0.30452029344 \Rightarrow 0.001 \leq 0.30452029344.$$

So, T is an F_α-contraction. Now, we will show that T is α-admissible. Note that

$$\alpha\left(Tx, Ty, Tz\right) = e^{\max\{0.1,0.1\}-0.1} = e^0 = 1.$$

So, T is αadmissible. Now, let $x_0 = 1 \in X = \mathbb{R}$.

$$\alpha(1,1,0.1) = e^{\max\{1,1\}-0.1} = e^{0.9} = 2.44 \geq 1.$$

Also, T is continuous, because $T(x,y,z) = 0.1$. So, T has a fixed point.

Next, we derive the following fixed point theorem for an $\alpha - \psi-$contraction in an S-metric space.

Definition 2.3. Let $T : X \to X$ and $\alpha, \eta : X \times X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η, if for all $x, y \in X$ such that $\alpha(x,x,y) \geq \eta(x,x,y)$, then we have $\alpha(Tx,Tx,Ty) \geq \eta(Tx,Tx,Ty)$. Note that if we take $\eta(x,x,y) = 1$, then T is called an α-admissible mapping. If we take $\alpha(x,x,y) = 1$, then T is called η-sub-admissible.

Theorem 2.2. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a mapping. Suppose that there exist two functions $\alpha, \eta : X \times X \times X \to [0,+\infty)$ such that T is α-admissible with respect to η. Let $r > 0$, $x_0 \in B(x_0, r)$ and $\psi \in \Psi$. Assume that*

$$x, y \in \overline{B(x_0,r)}, \alpha(x,x,y) \geq \eta(x,x,y) \Rightarrow S(Tx,Tx,Ty) \leq \psi S(x,x,y) \qquad (9)$$

and

$$2\sum_{i=0}^{j} \psi^i \left(S\left(x_0, x_0, Tx_0\right)\right) \leq r \text{ for all } j \in \mathbb{N}. \qquad (10)$$

Suppose that the following assertions hold:

(i) $\alpha(x_0,x_0,Tx_0) \geq \eta(x_0,x_0,Tx_0)$.
(ii) *For any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_n,x_{n+1}) \geq \eta(x_n,x_n,x_{n+1})$ for all $n \in N \cup \{0\}$ and $\{x_n\} \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$, then $\alpha(x_n,x_n,u) \geq \eta(x_n,x_n,u)$ for all $n \in N \cup \{0\}$.*

Then there is a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Tx^*$.

Proof. Let $x_0 \in X$ be such that

$$x_1 = Tx_0, \; x_2 = Tx_1 = T\left(Tx_0\right) = T^2 x_0.$$

Continuing in this way, we get

$$x_{n+1} = Tx_n.$$

By assumption,

$$\alpha\left(x_0, x_0, x_1\right) \geq \eta\left(x_0, x_0, x_1\right)$$

and as T is α-admissible with respect to η, so we have

$$\alpha\left(Tx_0, Tx_0, Tx_1\right) \geq \eta\left(Tx_0, Tx_0, Tx_1\right).$$

From which we can deduce that

$$\alpha\left(x_1, x_1, x_2\right) \geq \eta\left(x_1, x_1, x_2\right),$$

which also implies that

$$\alpha\left(Tx_1, Tx_1, Tx_2\right) \geq \eta\left(Tx_1, Tx_1, Tx_2\right).$$

Continuing in this way, we get

$$\alpha\left(x_n, x_n, x_{n+1}\right) \geq \eta\left(x_n, x_n, x_{n+1}\right) \text{ for all } n \in N \cup \{0\}.$$

First, we will show that $x_n \in \overline{B(x_0, r)}$, for all $n \in \mathbb{N}$. Using inequality (10) we have

$$S\left(x_0, x_0, Tx_0\right) \leq r.$$

It follows that

$$x_1 \in \overline{B(x_0, r)}.$$

Let $x_1, ...x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. If $j = 2i + 1$, where $i = 0, 1, 2...\frac{j-1}{2}$ then using in equality (9), we obtain

$$\begin{aligned} S(x_{2i+1}, x_{2i+1}, x_{2i+2}) &= S\left(Tx_{2i}, Tx_{2i}, Tx_{2i+1}\right) \\ &\leq \psi\left(x_{2i-1}, x_{2i-1}, x_{2i}\right) \\ &\leq \psi^2\left(S\left(x_{2i-2}, x_{2i-2}, x_{2i-1}\right)\right) \\ &\vdots \\ &\leq \psi^{(2i+1)} S\left(x_0, x_0, x_1\right). \end{aligned}$$

Thus, we have

$$S(x_{2i+1}, x_{2i+1}, x_{2i+2}) \leq \psi^{(2i+1)} S\left(x_0, x_0, x_1\right). \tag{11}$$

If $j = 2i + 2$, then as $x_1, x_2, ..., x_j \in \overline{B(x_0, r)}$, where $(i = 0, 1, 2, ..., \frac{j-2}{2})$, we obtain

$$S\left(x_{2i+2}, x_{2i+2}, x_{2i+3}\right) \leq \psi^{2(i+1)} S\left(x_0, x_0, x_1\right). \tag{12}$$

Thus, from inequalities (11) and (12) we have,

$$S\left(x_j, x_j, x_{j+1}\right) \leq \psi^j S\left(x_0, x_0, x_1\right). \tag{13}$$

Now

$$\begin{aligned} S\left(x_0, x_0, x_{j+1}\right) &\leq 2S\left(x_0, x_0, x_1\right) + 2S\left(x_1, x_1, x_2\right) + ... + 2S\left(x_j, x_j, x_{j+1}\right) \\ &\leq 2\sum_{i=0}^{j} \psi^i\left(S\left(x_0, x_0, x_1\right)\right) \\ &\leq r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$, for all $n \in \mathbb{N}$. Now, inequality (13) can be written as

$$S\left(x_n, x_n, x_{n+1}\right) \leq \psi^n S(x_0, x_0, x_1), \tag{14}$$

for all $n \in \mathbb{N}$.

Fix $\varepsilon > 0$ and let $n\left(\varepsilon\right) \in \mathbb{N}$ such that

$$\sum \psi^n\left(S\left(x_0, x_0, x_1\right)\right) < \varepsilon.$$

Let $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$, then by using the triangular inequality, we obtain

$$\begin{aligned} S(x_n.x_n, x_m) &\leq 2\sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1}) \\ &\leq 2\sum_{k=n}^{m-1} \psi^k (S(x_0, x_0, x_1)) \\ &\leq 2\sum_{k\geq n(\varepsilon)} \psi^k (S(x_0, x_0, x_1)) \\ &\leq \varepsilon. \end{aligned}$$

Thus, we proved that $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, S)$. As every closed ball in a complete S-metric space is complete, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$. Also

$$\lim_{n\to\infty} S(x_n, x_n, x^*) = 0. \tag{15}$$

On the other hand from (ii), we have

$$\alpha(x^*, x^*, x_n) \geq \eta(x^*, x^*, x_n) \text{ for all } n \in N \cup \{0\}. \tag{16}$$

Now, using the triangular inequality, together with (9) and (16), we get

$$\begin{aligned} S(Tx^*, Tx^*, x_{2i+1}) &\leq \psi(S(x^*, x^*, x_{2i})) \\ &\leq S(x^*, x^*, x_{2i}). \end{aligned}$$

So, we obtain that $S(Tx^*, Tx^*, x^*) = 0$, that is, $Tx^* = x^*$. Hence, T have a fixed point in $\overline{B(x_0, r)}$. □

If we take $\eta(x, x, y) = 1$, for all $x, y \in X$, in the above result, we obtain the following result.

Corollary 2.3. *Let (X, S) be a complete S-metric space, $T : X \to X$, $r > 0$ and x_0 be an arbitrary point in $\overline{B(x_0, r)}$. Suppose that there exists $\alpha : X \times X \times X \to [0, +\infty)$ such that T is α-admissible. For $\psi \in \Psi$, assume that*

$$x, y \in \overline{B(x_0, r)}, \alpha(x, x, y) \geq 1 \Rightarrow S(Tx, Tx, Ty) \leq \psi(S(x, x, y)) \tag{17}$$

and

$$2\sum_{i=0}^{j} \psi^i (S(x_0, x_0, Tx_0)) \leq r \text{ for all } j \in \mathbb{N}. \tag{18}$$

Suppose that the following assertions hold:

(i) $\alpha(x_0, x_0, Tx_0) \geq 1$.
(ii) *For any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $\{x_n\} \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$, then $\alpha(x_n, x_n, u) \geq 1$ for all $n \in N \cup \{0\}$.*

Then there is a point x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Tx^*$.

Corollary 2.4. *Let* (X, S) *be a complete* S*-metric space and* $T : X \to X$ *be a mapping. Suppose that there exists* $\eta : X \times X \times X \to [0, +\infty)$ *such that* T *is* η*-sub-admissible. Let* $\psi \in \Psi$ *and* x_0 *be an arbitrary point in* $\overline{B(x_0, r)}$. *Assume that*

$$x, y \in \overline{B(x_0, r)}, \eta(x, x, y) \leq 1 \Rightarrow S(Tx, Tx, Ty) \leq \psi(S(x, x, y))$$

and

$$2 \sum_{i=0}^{j} \psi^i (S(x_0, x_0, Tx_0)) \leq r \text{ for all } j \in \mathbb{N}.$$

Suppose that the following assertions hold:

(i) $\eta(x_0, x_0, Tx_0) \leq 1$.

(ii) *For any sequence* $\{x_n\}$ *in* $\overline{B(x_0, r)}$ *such that* $\eta(x_n, x_n, x_{n+1}) \leq 1$ *for all* $n \in N \cup \{0\}$ *and* $\{x_n\} \to u \in \overline{B(x_0, r)}$ *as* $n \to +\infty$ *then* $\eta(x_n, x_n, u) \leq 1$ *for all* $n \in N \cup \{0\}$.

Then there is a point x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Tx^*$.

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