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On character amenability of weighted convolution algebras on certain semigroups

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ABSTRACT. In this work, we study the character amenability of weighted convolution algebras $\ell^1(S,\omega)$, where S is a semigroup of classes of inverse semigroups with a uniformly locally finite idempotent set, inverse semigroups with a finite number of idempotents, Clifford semigroups and Rees matrix semigroups. We show that for inverse semigroup with a finite number of idempotents and any weight ω, $\ell^1(S,\omega)$ is character amenable if each maximal semigroup of S is amenable. Then for a commutative semigroup S and $\omega(x) \geq 1$, for all $x \in S$. Moreover, we show that character amenability of $\ell^1(S,\omega)$ implies that S is a Clifford semigroup. Finally, we investigate the character amenability of the weighted convolution algebra $\ell^1(S,\omega)$, and its second dual for a Rees matrix semigroup.

1. Introduction

Let A be a Banach algebra and E be a Banach A-bimodule. We regards the dual space E^* as a Banach A-bimodule with the following module actions:

$$(a.f)(x) = f(x.a)\ ,\ (f.a)(x) = f(a.x)\ (a \in A, f \in E^*, x \in E).$$

The notion of φ-amenability for Banach algebras was introduced by Kaniuth, Lau and Pym in [**11, 12**], where $\varphi : A \longrightarrow \mathbb{C}$ is a character. Monfared in [**18**] introduced the notion of character amenability for Banach algebras and some interesting results are given in [**19**]. Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \longrightarrow \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A in to $\mathbb{C}$, and let Φ_A denote the character space of A (that is, the set of all character on A). Approximate character amenability was introduced by Aghababa, Shi and Wu in [**1**] and Jabbari in [**8**], defined by characters on A, see [**18, 19**], for more details. Moreover, the character amenability of some versions of group algebras is investigated in [**9**]. These notions have been studied for various classes of Banach algebras, see [**5, 11, 12**],

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for more details. Forasmuch as character amenability is weaker than the classical amenability introduced by Johnson in [10], so all amenable Banach algebras are character amenable.

Module character amenability of Banach algebras which defines the notion of invariant functional concerning a Banach bimodule with compatible actions and applications to the semigroup algebras of an inverse semigroup is also introduced in [2]. It is shown in [13], that the character amenability of semigroup algebra $\ell^1(S)$ implies that the semigroup S is amenable and the authors focus on certain semigroups such as inverse semigroup, Rees semigroup, Clifford semigroup and Brandt semigroup and study the character amenability of $\ell^1(S)$ concerning the semigroup S.

Also in [22], Soroushmehr described the amenability of the weighted convolution algebra $\ell^1(S,\omega)$, where S is a regular Rees matrix semigroup and $\omega \geq 1$. No much work has been done to date on the character amenability version for weighted convolution algebra $\ell^1(S,\omega)$ on a semigroup S, as in the other notions for amenability. So this motivated us to see how the character amenability of $\ell^1(S,\omega)$ affects the structure of S. Thus, in this work, we study the character amenability of weighted convolution algebras on certain semigroups.

2. Preliminaries

We recall some standard notions from [3, 4]. Let A be a Banach algebra and E be a Banach A-bimodule. A continuous linear operator $D : A \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a).b + a.D(b)$, for all $a, b \in A$. Given $x \in E$, the *inner derivation* $ad_x : A \longrightarrow E$ is defined by $ad_x(a) = a \cdot x - x \cdot a$, for all $a \in A$. According to the Johnson's original definition, a Banach algebra A is called *amenable*, if for every Banach A-bimodule E, every derivation from A into E^* (the dual of E) is inner. The concept of amenability introduced by B. E. Johnson in [10]. Let A be a Banach algebra, and let X be a Banach A-bimodule, we let $M^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by

$$x.a = \varphi(a)x \ (a \in A \ , \ x \in X \ , \ \varphi \in \Phi_A),$$

and $M^A_{\varphi_l}$ denote the class of Banach A-bimodule X for which the left module action of A on X is given by

$$a.x = \varphi(a)x \ (a \in A \ , \ x \in X \ , \ \varphi \in \Phi_A).$$

It is easy to see that the left module action of A on the dual module X^* is given by

$$a.f = \varphi(a)f \ (a \in A \ , \ f \in X^* \ , \ \varphi \in \Phi_A).$$

Thus, we note that $X \in M^A_{\varphi_r}$ (resp. $X \in M^A_{\varphi_l}$) if and only if $X^* \in M^A_{\varphi_l}$ (resp .$X^* \in M^A_{\varphi_r}$). Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [19, 18] that

(i) A is *left φ-amenable* if every continuous derivation $D : A \longrightarrow X^*$ is inner for every $X \in M^A_{\varphi_r}$;
(ii) A is *right φ-amenable* if every continuous derivation $D : A \longrightarrow X^*$ is inner for every $X \in M^A_{\varphi_l}$;
(iii) A is *left character amenable* if it is left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is *right character amenable* if it is right φ-amenable for every $\varphi \in \Phi_A$;
(v) A is *character amenable* if it is both left and right character amenable.

We also recall that a *semigroup* is a non-empty set S with an associative binary operation $(s,t) \longrightarrow st$, $S \times S \longrightarrow S$ $(s,t \in S)$. Let S be a semigroup, S is said to be *regular* if for all $s \in S$, there is $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. S is an *inverse semigroup* if such s^* exists and is unique for all $s \in S$. An element $p \in S$ is *idempotent* if $p^2 = p$. The set of idempotents in S is denoted by $E(S)$. A semigroup S is *semilattice* if S is commutative and $E(S) = S$.

Let S be a semigroup. The semigroup algebra $\ell^1(S)$ is the completion in the ℓ^1-norm of the algebra $\mathbb{C}S$, the Banach algebra generated by the semigroup S. For $s \in S$, we write $\delta_s = \chi_{\{s\}}$ for the indicator function of the set $\{s\}$. The convolution product $*$ on $\ell^1(S)$ is uniquely defined by requiring that $\delta_s * \delta_t = \delta_{st}$ $(s,t \in S)$. There is always a character on the Banach algebra $\ell^1(S)$ that is the augmentation character $\varphi_S :\ \ell^1(S) \longrightarrow \mathbb{C}$ such that $f \mapsto f(s)$ $s \in S$.

Let S be a semigroup. A continuous function $\omega : S \longrightarrow (0,\infty)$ is a *weight* on S if $\omega(st) \leq \omega(s)\omega(t)$, for all $s,t \in$ S and $\Omega(g) := \omega(g)\omega(g^{-1})$. Then

$$\ell^1(S,\omega) = \{f = \Sigma_{s\in S} f(s)\delta_s : ||f||_\omega = \Sigma_{s\in S}|f(s)|\omega(s) < \infty\},$$

with $\|\cdot\|_\omega$ as a norm, is a Banach algebra which is called *weighted convolution algebra*.

3. Main results

In this section, we will consider the character amenability properties of weighted convolution algebras. First, we need the following results:

Theorem 3.1. [6, Theorem 2.3] *Let S be a semigroup and ω be a weight on S.*

(i) *If $\omega \geq 1$ and $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S)$ is character amenable.*
(ii) *If $\omega \leq 1$ and $\ell^1(S)$ is character amenable, then $\ell^1(S,\omega)$ is character amenable.*

Corollary 3.2. [6, Corollary 2.5] *Let $S = M^0(G,I)$ be the Brandt semigroup and ω be a weight on S. Then the following are equivalent:*

(i) *$\ell^1(S,\omega)$ is character amenable.*
(ii) *$\ell^1(S)$ is character amenable.*

(iii) *I is finite and in the case where $|I| = 1$, then G is amenable.*

Using our main result, we extend some results of [13], to weighted convolution algebras.

Proposition 3.3. *Let S be a semigroup, ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then S is amenable and regular.*

Proof. Since $\ell^1(S,\omega)$ is character amenable, by Theorem 3.1, $\ell^1(S)$ is character amenable, so by [13, Proposition 4.1(ii)], S is amenable and regular, as required. □

Corollary 3.4. *Let S be a semigroup with $E(S)$ finite, ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then it has an identity.*

Proof. By Proposition 3.3, S is regular and amenable. Thus from finiteness of $E(S)$, there is a finite subset $F \subset E(S)$ such that

$$S = \cup\{pSq \ : \ p, q \in F\}.$$

Set $A = \ell^1(S,\omega)$. There exist $m \in \mathbb{N}$, $p_1, ..., p_m \in F$, and pairwise disjoint subsets T_i of S, for any $i \in \mathbb{N}_m$ such that $T_i \subset p_i S$ $(i \in \mathbb{N}_m)$ and $S = \cup\{T_i \ : \ i \in \mathbb{N}_m\}$. For each $f \in A$ and $i \in \mathbb{N}_m$, we have $f|T_i = (\delta_{p_i} \star f)|T_i$. Since $A = \ell^1(S,\omega)$, is character amenable, by [11, Proposition 1(i)], A has a bounded approximate identity. So A has a left approximate identity and from finiteness of F there is a sequence (f_n) in A such that

$$\|f_n \star \delta_p - \delta_p\|_1 < \frac{1}{n} \ (n \in \mathbb{N}, p \in F). \tag{1}$$

We claim that (f_n) is a Cauchy sequence. Take $\lambda \in (A^*)_{[1]} = \ell^\infty(S, \frac{1}{\omega})_{[1]}$, and for $i \in \mathbb{N}_m$, set $\lambda_i = \lambda|T_i$, so that $\lambda_i \in (A^*)_{[1]}$. Clearly, we have $\lambda = \sum_{i=1}^m \lambda_i$. For $k < n$ and $i \in \mathbb{N}_m$, we have

$$|\langle f_k - f_n, \lambda_i\rangle| = |\langle \delta_{p_i} \star (f_k - f_n), \lambda_i\rangle| \leq \frac{2}{k},$$

and so $|\langle f_k - f_n, \lambda\rangle| \leq \dfrac{2m^2}{k}$. Thus $\|f_k - f_n\|_1 \leq \dfrac{2m^2}{k}$, giving the claim set $f = Lim_{n\longrightarrow\infty} f_n \in A$, and take $i \in \mathbb{N}_m$ and $t \in T_i$. Then, by (1), we have

$$f \star \delta_t = \lim_{n\to\infty} f_n \star \delta_{p_i} \star \delta_t = \delta_{p_i} \star \delta_t = \delta_t.$$

Since $S = \bigcup\{T_i : i \in \mathbb{N}_m\}$ it follows that f is a left identity of A. Similarly A has a right identity, and $\ell^1(S,\omega)$ has an identity. □

A semigroup S is called *left cancellative* if, for all $a, x, y \in S$, $ax = ay$ implies that $x = y$.

Corollary 3.5. *Let S be a left cancellative semigroup, ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then S is an amenable group.*

Proof. By Proposition 3.3, S is amenable and regular. Since S is regular, it follows that, for each $s \in S$, there exists $e_s \in E(S)$ such that $se_s = s$. Since S is left cancellative, the element e_s is uniquely defined by this equation. Since S is amenable, it is left reversible [20, Proposition (1.23)]; this means that, for each pair $\{s,t\}$ in S, there exists $x \in sS \cap tS$, say $x = sy = tz$ for some $y, z \in S$. Clearly $yse_x = ys$ and so $se_x = s$, because S is left cancellative. Thus $e_x = e_s$. Similarly $e_x = e_t$, and so $e_s = e_t$. Thus there is a unique element $e \in S$ such that $se = s$ $(s \in S)$.

Let $s \in S$. Then $e^2 s = es$, and so $es = s$, again by left cancellativity. Thus e is the identity of S. Take $s \in S$. By the regularity of S, there exists $t \in S$ with $sts = s$. By replacing t by sts we may suppose that also $tst = t$. We have $ts = st = e$ by left cancellativity, and so $s = t^{-1} \in S$. Thus S is a group. □

Theorem 3.6. *Let S be an inverse semigroup with $E(S)$ finite and ω be a weight on S. If each maximal semigroup of S is amenable, then $\ell^1(S, \omega)$ is character amenable.*

Proof. Since $E(S)$ is finite and S is inverse, S has a principal series

$$S = S_1 \supset S_2 \supset S_3 \supset .... \supset S_{m-1} \supset S_m = K(S)$$

of ideals of S, where $K(S)$ is the minimum ideal, see [4, Theorem 3.12].*Thus*, $\frac{S_i}{S_{i+1}}$ is a simple inverse semigroup with a finite number of idempotents, and so is a group. Also, for $i = 1, 2, ..., n-1$, $\frac{S_i}{S_{i+1}}$ is 0-simple with a finite number of idempotents, and so is a completely 0-simple inverse semigroup, that is a Brandt semigroup. By Corollary 3.2, $\ell^1(S, \omega)$ is character amenable if and only if $\ell^1(S)$ is character amenable and by proof of [13, Proposition 3.1], $\ell^1(S)$ is character amenable if and only if $\ell^1(\frac{S_i}{S_{i+1}})$ is character amenable for $i = 1, 2, ..., n-1$. For $i = 1, 2, ..., n-1$, let G_i be the group of the Brandt semigroup $\frac{S_i}{S_{i+1}}$ and $\ell^1(\frac{S_i}{S_{i+1}})$ is amenable if G_i is amenable for $i = 1, 2, ..., n-1$. So $\ell^1(S, \omega)$ is character amenable if G_i is amenable and the groups G_i are maximal subgroups of S. □

For an inverse semigroup S and $p \in E(S)$, we set

$$G_p = \{s \in S\ ;\ ss^{-1} = s^{-1}s = p\}.$$

Then G_p is a group with identity p. It is called the maximal subgroup of S at p. We recall that a Clifford semigroup is an inverse semigroup S for which $ss^{-1} = s^{-1}s$ $(s \in S)$. For a Clifford semigroup S, we have $s \in G_{ss^{-1}}$, and so S is a disjoint union of the groups G_p $(p \in E(S))$, see [7], for more details.

Corollary 3.7. *Let $S = \cup_{p\in E(S)} G_p$ be a Clifford semigroup such that $E(S)$ is finite and ω be a weight on S. Then $\ell^1(S, \omega)$ is character amenable if G_p is amenable for each $p \in E(S)$.*

Proof. This follows from Theorem 3.6. □

The following example shows that finiteness of $E(S)$ is necessary.

Example 3.1. Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite and each G_e is amenable, ω be a weight on S and $\omega \geq 1$. Then the weighted convolution algebra $\ell^1(S,\omega)$ is not character amenable if $E(S)$ is not finite; If $\ell^1(S,\omega)$ is character amenable, by hypothesis and theorem 3.1, $\ell^1(S)$ is character amenable. But since

$$\ell^1(S) \cong \ell^1 - \bigoplus_{e\in E(S)} \ell^1(G_e).$$

(see [21, Theorem 2.16] and [1, Proposition 6.3]), $\ell^1(S)$ is not character amenable, by [1, Proposition 6.3], and this is a contradiction.

Theorem 3.8. *Let S be a commutative semigroup. Let ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then S is a Clifford semigroup.*

Proof. By Proposition 3.3, S is regular. A commutative regular semigroup is an inverse semigroup which is a semilattice of abelian group. Thus $S = \cup_{\alpha\in Y} S_\alpha$, is a Clifford semigroup. □

Corollary 3.9. *Let S be a commutative semigroup such that $E(S)$ is finite and let ω be a weight on S and $\omega \geq 1$. Then the following statements are equivalent:*

(i) *$\ell^1(S,\omega)$ is character amenable;*
(ii) *S is a Clifford semigroup.*

Proof. By Theorem 3.8, the implication $(i) \longrightarrow (ii)$, is clear.
$(ii) \longrightarrow (i)$ Let S be a Clifford semigroup, indeed, as S is a commutative Clifford semigroup, each maximal subgroup of S is commutative, and therefore it is amenable. So, by Theorem 3.6, $\ell^1(S,\omega)$ is character amenable. □

Let P be a partially ordered set. For $p \in P$, we define $(p] = \{x : x \leq p\}$ and $[p) = \{x : p \leq x\}$. Then P is locally finite if $(p]$ is finite, for each $p \in P$, and P is locally C-finite, for some constant $C \geq 1$, if $|(p]| < C$, for each $p \in P$. A partially ordered set that is locally C-finite for some C is uniformly locally finite.

Let S be an inverse semigroup. Then S is [locally finite/ C-locally finite/ uniformly locally finite] respectively if the partially ordered set $(E(S), \leq)$ has the corresponding property, see [21], for more details.

Proposition 3.10. *Let S be a inverse semigroup such that $(E(S), \leq)$ is uniformly locally finite and ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then each maximal subgroup of S is amenable.*

Proof. Let $\ell^1(S,\omega)$ be character amenable, then by Theorem 3.1, $\ell^1(S)$ is character amenable. Since $(E(S),\leq)$ is uniformly locally finite, $(S,\leq)$ is uniformly locally finite by [**21**, Proposition 2.14] and now using [**21**, Theorem 2.18], we have

$$\ell^1(S)\cong l^1-\bigoplus\{\mathbb{M}_{E(D_\alpha)}(\ell^1(G_{p_\alpha}))\ :\ \alpha\in J\},$$

and so, for each $\alpha\in J$, $\mathbb{M}_{E(D_\alpha)}(\ell^1(G_{p_\alpha}))$ is a homomorphic image of $\ell^1(S)$. Then by [**18**, Theorem 2.6(i)], we have

$$\mathbb{M}_{E(D_\alpha)}(\ell^1(G_{p_\alpha}))\cong\mathbb{M}_{E(D_\alpha)}(C)\otimes(\ell^1(G_{p_\alpha}))$$

is character amenable for each $\alpha\in J$. Thus $\mathbb{M}_{E(D_\alpha)}(\ell^1(G_{p_\alpha}))$ is left character amenable. Moreover, $\ell^1(G_{p_\alpha})$ is left character amenable by [**13**, Corollary 3.3]. So using [**18**, Corollary 2.4], $\ell^1(G_{p_\alpha})$ is left character amenable if and only if G_{p_α} is an amenable group. □

4. Weighted Rees matrix semigroup algebras

In this section, we give results on weighted Rees semigroup algebras. Rees semigroups are described in [**4, 7, 17, 14**]. Indeed, let G be a group, $m,n\in N$, and $G^0=G\cup\{0\}$. Let

$$S=\{(g)_{ij}\ :\ g\in G,1\leq i\leq m,1\leq j\leq n\}\cup\{0\},$$

where $(g)_{ij}$ denotes the element of $M_{m\times n}(G^0)$ with g in the $(i,j)^{th}$ place and 0 elsewhere and 0 is a matrix with 0 everywhere. Let $P=(p_{ji})$ be an $n\times m$ matrix over G^0. Then the set S with the composition $(g)_{ij}\circ 0=0\circ(g)_{ij}=0$ and $(g)_{ij}\circ(h)_{lk}=(gp_{jl}h)_{ik}$, $((g)_{ij},(h)_{lk}\in S)$ forms a semigroup which is called a *Rees matrix semigroup* with a zero over G, and it will be denoted by $S=M^0(G,P,m,n)$. The matrix P is called the *sanwich matrix* in each case. We write $S=M^0(G,P,n)$ for $S=M^0(G,P,n,n)$ in this case where $m=n$.

The above sandwich matrix P is *regular* if every row and column contains at least one entry in G; the semigroup $S=M^0(G,P,m,n)$ is regular as a semigroup if and only if the sandwich matrix is regular.

In [**4**], the Rees matrix semigroup algebra $\ell^1(S)$ is described as follows: for $g\in G$, $(g)_{ij}$ is identified with the element of $M_{m\times n}(\ell^1(G))$ which has δ_g in the $(i,j)^{th}$ place and 0 elsewhere, and $\circ$ is identified with δ_0. Furthermore, $P\in M_{n\times m}(G^0)$ is identified with a matrix $P\in M_{n\times m}(\ell^1(G))$ as follows: if the initial matrix P has $g\in G$ in the $(i,j)^{th}$-position, then the new matrix P has the point mass δ_g in the $(i,j)^{th}$-position; if the first matrix P has 0 in the $(i,j)^{th}$-position, then the new matrix P has 0 in the $(i,j)^{th}$-position. Using this identification, it is shown that $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is isometrically isomorphic to the Munn algebra $M(\ell^1(G),P,m,n)$, where $\mathbb{C}\delta_0$ is a one-dimensional ideal. $\frac{\ell^1(S)}{\mathbb{C}\delta_0}=M(\ell^1(G),P,m,n)$, is unital. With $m=n$, since $M(\ell^1(G), P, n, n) = M(\ell^1(G), P, n)$, is also unital and so the Munn algebra $M(\ell^1(G), P, n)$, is topologically isometric to $M_n(\ell^1(G))$, see [4], for more details.

Let S be completely 0-simple with finitely many idempotents, and let ω be a weight on S (not necessary greater than 1). Then there is a maximal subgroup G of S such that

$$S \simeq M^0(G, P, m, n),$$

and

$$\frac{\ell^1(S,\omega)}{\mathbb{C}\delta_0} \simeq M(\ell^1(G,\omega), P, m, n),$$

see [22, Theorem 2.1], for more details. Let G be a group, and let ω be a weight on G. A weight on G is said to be *symmetric* if $\omega(t^{-1}) = \omega(t)$, for every $t \in G$.

The following result is very useful in the proof of our main result in this section and it's proof follows from [6, Theorem 2.4].

Theorem 4.1. *Let S be a semigroup with a zero element and ω be a weight on S. If $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S)$ is character amenable.*

Theorem 4.2. *Let $S = M^0(G, P, I, J)$ and ω be a symmetric weight on S. Then the following statements are equivalent:*

(i) *$\ell^1(S,\omega)$ is character amenable.*
(ii) *$\ell^1(G,\omega)$ is character amenable, $|I| = |J| < \infty$ and P is invertible.*
(iii) *$\ell^1(S)$ is character amenable and Ω is bounded on G.*

Proof. $(i) \longrightarrow (ii)$ Let $\ell^1(S,\omega)$ be character amenable. By Theorem 4.1, $\ell^1(S)$ is character amenable and so is left character amenable. Then by [13, Theorem 4.11], $\ell^1(S)$ is amenable. Hence, by [4], $|I| = |J| = n < \infty$ and P is invertible and the equality

$$\frac{\ell^1(S,\omega)}{\mathbb{C}\delta_0} \simeq M(\ell^1(G,\omega), P, n)$$

shows that $M(\ell^1(G,\omega), P, n)$ is character amenable, by [15, Proposition 3.1]. Then $\ell^1(G,\omega)$ is character amenable, by [13, Corollary 3.3].

$(ii) \longrightarrow (iii)$ Suppose that $\ell^1(G,\omega)$ is character amenable. By [16, Corollary 5], $\ell^1(G,\omega)$ is amenable and by [16, Proposition 4], G is amenable and Ω is bounded on G.

Since $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is isometrically isomorphic to the Munn algebra $\mathbb{M}_n(\ell^1(G))$, and amenability of G shows that $\mathbb{M}_n(\ell^1(G))$ is amenable, see [10]. Then $\ell^1(S)$ is amenable and so it is character amenable, as required.

$(iii) \longrightarrow (i)$ Let $\ell^1(S)$ be character amenable. By [13, Theorem 4.11], $\ell^1(S)$ is amenable and the amenability of $\ell^1(S)$ implies that $\dfrac{\ell^1(S)}{\mathbb{C}\delta_0} \simeq \mathbb{M}_n(\ell^1(G))$, where $|I| = |J| = n$. Thus, $\mathbb{M}_n(\ell^1(G))$ is amenable, and so G is amenable. Amenability of G with the boundedness of ω on G implies that $\ell^1(G,\omega)$ is amenable. We recall that $\dfrac{\ell^1(S,\omega)}{\mathbb{C}\delta_0} \simeq \mathbb{M}_n(\ell^1(G,\omega))$ and by [**21**, Theorem 2.7], $\mathbb{M}_n(\ell^1(G,\omega))$ is amenable, then $\ell^1(S,\omega)$ is amenable, so $\ell^1(S,\omega)$ is character amenable, as required. □

Corollary 4.3. *Let $S = M^0(G,P,n)$ be a Rees matrix semigroup with a zero over the group G, sandwich matrix P and ω be a weight on S. Then $\ell^1(S,\omega)$ is character amenable if and only if it is amenable.*

Proof. Suppose that $\ell^1(S,\omega)$ is character amenable, then by Theorem 4.2, $\ell^1(S)$ is character amenable and Ω is bounded on G. Since $\ell^1(S)$ is character amenable, $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is character amenable by [**15**, Proposition 3.1]. Also, since $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is isomorphic to $\mathbb{M}_n(\ell^1(G))$, $\mathbb{M}_n(\ell^1(G))$ is character amenable and so $\mathbb{M}_n(\ell^1(G))$ is left character amenable. Hence, by [**13**, Proposition 3.4], $\mathbb{M}_n(\ell^1(G))$ is amenable and so $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is amenable. Then $\ell^1(S)$ is amenable. Now, by [**22**, Theorem 3.6], $\ell^1(S,\omega)$ is amenable. The converse is clear. □

Notation 4.1. Let S be a semigroup, I be an ideal of S and ω be a weight on S. For $s,t \in S$, set $s \sim t$ either if $s = t$ or $s,t \in I$. Clearly, $\sim$ is an equivalence relation on S; the equivalence class containing s is denoted by $[s]$. Let $s,t \in S$ and define $[s][t] = [st]$. Evidently, this gives a well-defined semigroup operation on the set of equivalence classes $S/\sim$. So one may form the quotient semigroup S/I with the zero element I. Moreover, the map $S \longrightarrow S/I$, $s \mapsto [s]$ is an epimorphism, see [**7**, **22**], for more details.

Define $\tilde{\omega} : S/I \longrightarrow \mathbb{C}$, Such that $\tilde{\omega}([s]) = 1$ for all $s \in I$ and $\tilde{\omega}([s]) = \omega(s)$ for all $s \in S - I$. It is easy to see that $\tilde{\omega}$ is a weight on S/I. Now, we need the following result.

Lemma 4.4. [**22**, Lemma 3.1] *Let S be a semigroup, I be an ideal of S and ω be a weight on S. Then $\ell^1_0(I,\omega)$ is an ideal of $\ell^1(S,\omega)$ and*

$$\ell^1(S/I,\tilde{\omega}) \cong \ell^1(S,\omega)/\ell^1_0(I,\omega);$$

in particular, when $S = I$,

$$\ell^1(S,\omega)/\ell^1_0(S,\omega) \simeq \mathbb{C}.$$

Lemma 4.5. *Let S be a semigroup, I be an ideal of S and ω be a weight on S.*

(i) *If $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S/I,\tilde{\omega})$ is character amenable.*
(ii) *If both $\ell^1(S/I,\tilde{\omega})$ and $\ell^1_0(I,\omega)$ are character amenable, then $\ell^1(S,\omega)$ is character amenable.*
(iii) *If $\ell^1(S,\omega)$ is character amenable and $\ell^1_0(I,\omega)$ has a bounded approximate identity, then $\ell^1_0(I,\omega)$ is character amenable.*

Proof. By [**23**, Theorem 3.1.1] and [**15**, Proposition 3.1] the proof is clear. □

Theorem 4.6. *Let S be a semigroup and ω be a symmetric weight on S. Then the following statements are equivalent:*

(i) *$\ell^1(S,\omega)$ is character amenable;*
(ii) *$\ell^1(S)$ is character amenable and Ω is bounded on every maximal subgroup G of S.*

Proof. $(i) \longrightarrow (ii)$ Let $\ell^1(S,\omega)$ be character amenable. By [4], S has a principal series

$$S_1 \trianglelefteq S_2 \trianglelefteq S_3 \trianglelefteq ..... \trianglelefteq S_{n-1} \trianglelefteq S_n = S.$$

such that each quotient S_{j+1}/S_j is a regular Rees matrix semigroup of the form $M^0(G_i, P_i, n_i)$, for each i, where $n_i \in \mathbb{N}$ and $S_1 \cup \{G_i \ : \ 2 \leq n\}$ is the set of all maximal subgroups of S. Furthermore, S_1 is an ideal subgroup of S. $\ell^1_0(S_1,\omega)$ is an ideal of $\ell^1(S,\omega)$ and $\ell^1(S/S_1,\tilde{\omega})$ are character amenable (see Lemma 4.5). Since S_1 is a group, $\ell^1(S_1,\omega)$ has a bounded approximate identity and by Lemma 4.5, $\ell^1(S_1,\omega)$ is character amenable. Since ω is symmetric, by [15, Proposition 5.3 (1)], $\ell^1(S_1,\omega)$ is amenable. Thus by [16, Proposition 4], S_1 is amenable group and Ω is bounded on S_1. By [22, Theorem 2.1], for $2 \leq i \leq n$, we have

$$\ell^1(S_{i+1}/S_i,\tilde{\omega}) \simeq M(\ell^1(G_i,\omega), P_i, n_i)/\mathbb{C}\delta_0.$$

Since $\ell^1(S_{i+1}/S_i,\tilde{\omega})$ is character amenable, $M(\ell^1(G_i,\omega), P_i, n_i)$ is character amenable and so $\ell^1(G_i,\omega)$ is character amenable. Now, by Theorem 4.2, $\ell^1(S)$ is character amenable and Ω is bounded on G_i. So, Ω is bounded on every maximal subgroup G on S.

$(ii) \longrightarrow (i)$ Let $\ell^1(S)$ be character amenable. By [13, Proposition 4.1(ii)], S is amenable. Hence, S_1 is amenable group. From boundedness of Ω on S_1, we have $\ell^1(S_1,\omega)$ is amenable. Then by the same reasons in the proof of [22, Theorem 3.6], $\ell^1(S,\omega)$ is amenable and so it is character amenable. □

Proposition 4.7. *Let $S = M^0(G,P,I,J)$ and ω be a weight on S. Then the following statements are equivalent:*

(i) *$\ell^1(S,\omega)^{**}$ is character amenable.*
(ii) *S is finite, $|I| = |J| = n$ and P is invertible.*
(iii) *$\ell^1(S)$ is character amenable and S is finite.*

Proof. $(i) \longrightarrow (ii)$ Let $\ell^1(S,\omega)^{**}$ is character amenable, by [15, Theorem 4.5], $\ell^1(S,\omega)$ is character amenable. The definition of Rees matrix semigroup, shows that S has a zero element, so by Theorem 4.1, $\ell^1(S)$ is character amenable. Now, by [13, Theorem 4.11], $\ell^1(S)$ is amenable. This shows that $|I| = |J| = n$ and P is invertible by [4]. By corollary 4.3, $\ell^1(S,\omega)$ is amenable. Now, by using similar argument in [22, Theorem 3.7], we show that S is finite.

$(ii) \longrightarrow (iii)$ Since S is finite, G is finite and so G is amenable. By Johnson's Theorem [10], $\ell^1(G)$ is amenable. Then $\mathbb{M}_n(\ell^1(G))$ is amenable, and this follows from the above isometric isomorphism

$$\frac{\ell^1(S)}{\mathbb{C}\delta_0} \simeq M(\ell^1(G), P, m, n).$$

Then $\ell^1(S)$ is amenable and so it is character amenable.

$(iii) \longrightarrow (i)$ The finiteness of S implies that, ω is bounded on the whole of S, and so, $\ell^1(S,\omega) \simeq \ell^1(S)$. Thus, $\ell^1(S)$ is finite-dimensional and $\ell^1(S) \simeq \ell^1(S,\omega)^{**}$, so $\ell^1(S,\omega)^{**}$ is character amenable by [15, Proposition 3.1], as required. □

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