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A proof of the Cauchy–Schwarz inequality from the change of reference frame

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Abstract. Inspired by [1] a proof of the Cauchy–Schwarz inequality is given by considering the transformation between two different inertial reference frames

Consider an N–body system of particles with masses m_i, and two inertial reference frames K and K' with relative velocity $\mathbf{V}$, then the velocities of each body i in the two frames are related by the transformation

$$\mathbf{v}_i = \mathbf{v'}_i + \mathbf{V} \ , \tag{1}$$

thus their momenta obey the relation

$$\mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v}_i = \sum_{i=1}^{N} m_i \mathbf{v'}_i + \sum_{i=1}^{N} m_i \mathbf{V} = \mathbf{P'} + \mathbf{V} \sum_{i=1}^{N} m_i \ . \tag{2}$$

The reference frame K' in which the total momentum $\mathbf{P'} = \mathbf{0}$ is the center of mass reference. In this frame from (2) one obtains that

$$\mathbf{V} = \frac{\mathbf{P}}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \mathbf{v}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \mathbf{v}_i}{M} \ . \tag{3}$$

In the center of mass frame the N–body system acts like a single object of mass M and velocity $\mathbf{V}$.

The total energy of the system depends on the reference frame chosen. As the potential energy does not depend on the velocity, only the kinetic energy is affected by the transformation (1), therefore we obtain the total energy in the two different

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reference frames K and K':

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i (\mathbf{v}'_i + \mathbf{V})^2 =$$

$$\frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}'^2_i + \mathbf{V}\cdot\sum_{i=1}^{N} m_i \mathbf{v}'_i + \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{V}^2 = E' + \mathbf{V}\cdot\mathbf{P}' + \frac{1}{2}M\mathbf{V}^2 \,. \tag{4}$$

In the center of mass frame the relation (4) simplifies to $E = E' + M\mathbf{V}^2/2$, that is

$$\frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}'^2_i + \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{V}^2 \,, \tag{5}$$

and since all terms are positive this leads to the inequality

$$\sum_{i=1}^{N} m_i \mathbf{v}_i^2 \geq \sum_{i=1}^{N} m_i \mathbf{V}^2 = M\mathbf{V}^2 \,. \tag{6}$$

Multiplying by $\sum_{i=1}^{N} m_i = M$ both terms and using (3) we obtain

$$\left(\sum_{i=1}^{N} m_i^2\right)\left(\sum_{i=1}^{N} \mathbf{v}_i^2\right) \geq M^2\mathbf{V}^2 = \left(\sum_{i=1}^{N} m_i \mathbf{v}_i\right)^2 \,, \tag{7}$$

giving another proof of the Cauchy–Schwarz inequality. The equality occurs when all particles are at rest in the K' frame, that is when $\mathbf{v}'_i = \mathbf{0}$ and $\mathbf{v}_i = \mathbf{V}$ for all $i = 1 \ldots N$.

Observe also that this proof does not require the conservation of energy of the system.

References

[1] T. Tokieda, *A viscosity proof of the Cauchy–Schwarz inequality*, Amer. Math. Month., **122**(8) (2015), p. 781.

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