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Some results on disjointness preserving Fredholm operators between certain Banach function algebras

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Abstract. For two algebras $\mathcal{A}$ and $\mathcal{B}$, a linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is disjointness preserving if $x \cdot y = 0$ implies $Tx \cdot Ty = 0$ for all $x, y \in \mathcal{A}$ and is said Fredholm if $\dim(\ker(T))$ i.e. the nullity of T and $\operatorname{codim}(T(E))$ i.e. the corank of T are finite. We develop some results of Fredholm linear disjointness preserving operators from $C_0(X)$ into $C_0(Y)$ for locally compact Hausdorff spaces X and Y in [9], into regular Banach function algebras. In particular, we consider weighted composition Fredholm operators as a typical example of disjointness preserving Fredholm operators on certain regular Banach function algebras.

1. Introduction

Let $\mathcal{A}, \mathcal{B}$ be two spaces of functions a map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is disjointness preserving if $f \cdot g = 0$ implies $Tf \cdot Tg = 0$ for all $f, g \in A$. Weighted composition operators are examples of linear disjointness preserving or separating operators between spaces of functions. When X and Y are compact Hausdorff spaces, each linear separating bijection operator $T : C(X) \longrightarrow C(Y)$ is a continuous weighted composition operator where $C(X)$ is the Banach algebra of all complex-valued functions on X with supremum norm [7]. This result has been extended to $C_0(X)$, the Banach algebra of all continuous complex valued function on locally compact space X, which is vanishing at infinity [8]. Linear operators $T : L_p(\mu) \longrightarrow L_p(\mu)$ with the property that $f \cdot g = 0$, a.e. implies $Tf \cdot Tg = 0$, a.e. were considered by Banach in [5]. Disjointness preserving operators between two vector lattices is studied in [1, 4]. It was proved in [6] that when $\mathcal{A}, \mathcal{B}$ are certain regular semisimple commutative Banach algebras then every separating bijection is automatically continuous and its inverse is separating and under extra conditions on $\mathcal{B}$ induced a homeomorphism between the structure spaces of $\mathcal{A}$ and $\mathcal{B}$. In their joint paper [9], J. Jeang and

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N.C. Wong considered Fredholm linear separating operators from $C_0(X)$ into $C_0(Y)$ for locally compact Hausdorff spaces X and Y and showed that if there exists such map, then X and Y are homeomorphic after removing finite subsets. A complete description of the Fredholm disjointness preserving operators between ultrametric spaces of (bounded and not necessarily bounded) continuous functions defined on N-compact spaces given in [3].

In this paper, we give some results about disjointness preserving Fredholm operators between certain regular Banach function algebras and in the sequel we will deal with properties of weighted composition Fredholm operators as a standard example of all disjointness preserving Fredholm operators.

2. preliminaries

Let $\mathcal{A}$ be a commutative Banach algebra, the space of all multiplicative linear functional on $\mathcal{A}$ which is called the structure space of $\mathcal{A}$, i.e. $\sigma(\mathcal{A})$, is a locally compact Hausdorff space with respect to Gelfand topology. For $a \in \mathcal{A}$, let $\hat{a} \in C_0(\sigma(\mathcal{A}))$ be its Gelfand transform of a such that $\hat{a}(\varphi) = \varphi(a)$, for all $\varphi \in \sigma(\mathcal{A})$. In this case if the Gelfand transform $a \to \hat{a}$ is injective then $\mathcal{A}$ is called semisimple. A commutative Banach algebra $\mathcal{A}$ is said to be regular if for each closed subset E of $\sigma(\mathcal{A})$ and $\varphi \in \sigma(\mathcal{A})\backslash E$ there exists $a \in \mathcal{A}$, such that $\hat{a}(\varphi) = 1$ and $\hat{a} = 0$ on E.

Let X be a locally compact Hausdorff space, a subalgebra $\mathcal{A}$ of $C_0(X)$ is called Banach function algebra, if it is separating the points of X and for all $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. It is clear that every Banach function algebra is commutative and semisimple and each commutative semisimple Banach algebra is considered, as a Banach function algebra on its structure space $\sigma(\mathcal{A})$. When $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach function algebra on a locally compact Hausdorff space X, then $\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{A}}$, where $\|\cdot\|_\infty$ is supremum norm of $C_0(X)$, also for each $f \in \mathcal{A}$ and $x \in X$ we use $f(x)$ instead of $\hat{f}(x)$. A uniform algebra on X is a Banach function algebra whose complete norm is the supremum norm on X. For each ideal I of $\mathcal{A}$ we denote the hull set of I by $h_{\mathcal{A}}(I) = \{x \in X : f(x) = 0, \text{forall} f \in I\}$. The Jacobson radical of a commutative Banach algebra $\mathcal{A}$ is defined by $\text{Rad}(\mathcal{A}) = \bigcap_{\varphi\in\sigma(\mathcal{A})} \ker\varphi$.

Let X_∞ be the one-point compactification of $X, \text{cl}_{X_\infty}(E)$ and $\text{int}(E)$ mean respectively the closure and interior of subset E of X in X_∞, $\text{coz}(f)$ denote the cozero set of $f \in \mathcal{A}$ i.e. the set $\{x \in X : f(x) \neq 0\}$.

A Banach function algebra $\mathcal{A}$ on a locally compact Hausdorff space X is said to satisfy Ditkin's condition, if for each $x \in X_\infty$ and $f \in \mathcal{A}$, with $f(x) = 0$ there exists a sequence $\{f_n\}$ in $\mathcal{A}$ such that f_n vanishing on a neighborhood of x and $\|f_n f - f\|_{\mathcal{A}} \longrightarrow 0$.

Let E and F be Banach vector spaces a linear map $T : E \longrightarrow F$ is said Fredholm if $\dim(\ker(T))$ i.e. the nullity of T and $\text{codim}(T(E))$ i.e. the corank of T are finite. We say that a bounded linear map $T : E \longrightarrow F$ is bounded bellow if there exists positive real number r such that $\|Te\| \geq r\|e\|$, for each $e \in E$. It follows from the open mapping theorem that T is a bounded bellow if and only if T is injective and has closed range. Also each bounded linear map T with finite corank has a closed range, see [**2**].

3. Fredholm operators between certain Banach function algebras

Suppose $\mathcal{A}$ and $\mathcal{B}$ be Banach function algebras on their structure spaces X and Y respectively and $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a disjointness preserving operator. The evaluation map δ_y on $\mathcal{B}$ is defined by $\delta_y(g) = g(y)$ for each $y \in Y$ and Y_0 will stand for the set of elements $y \in Y$ where $\delta_y \circ T \neq 0$. In this case, for each $y \in Y_0$, the support of $\delta_y \circ T$ denoted by supp($\delta_y \circ T$) , is defined as the set of all $x \in X_\infty$ the one point compactification of X such that for each neighborhood U of x in X_∞ there exists an element $g \in B$ with $\mathrm{coz}(g) \subseteq U$ and $\delta_y \circ T(g) \neq 0$. Let $y \in Y_0$ then the set supp($\delta_y \circ T$) is non empty. If, in addition, A is regular, then supp($\delta_y \circ T$) is a singleton (see Lemma 1 of [**6**]). In this case the support map $h : Y_0 \longrightarrow X$ of T is defined as $h(y) =$supp$(\delta_y \circ T)$. In the sequel we shall use the following proposition concerning disjointness preserving operators between regular Banach function algebras.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be regular Banach function algebras with structure spaces X and Y, respectively, such that $\mathcal{A}$ satisfies the Ditkin' s condition. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a disjointness preserving operator. There exist continuous maps $h : Y_0 \longrightarrow X$ (which is called the support map of T) and $\omega : h^{-1}(X) \subseteq Y_0 \longrightarrow \mathbb{C}$ which is non-vanishing such that*

(a) for each neighborhood U in X_∞ and $f \in \mathcal{A}$, $f_{|U\cap X} \equiv 0$ implies that $Tf_{|h^{-1}(U)} \equiv 0$.

(b) $h(\mathrm{coz}(Tf)) \subseteq \mathrm{cl}_{X_\infty}(\mathrm{coz}(f))$ for all $f \in \mathcal{A}$.

(c) Let Y_c be the set of all $y \in Y_0$, such that $\delta_y \circ T$ is continuous on $(\mathcal{A}, \| \cdot \|_{\mathcal{A}})$ and Y_d be the complement of Y_c in Y_0, then $h(Y_c) \subseteq X$.

(d) For each $y \in Y$, the equality $Tf(y) = \omega(y) \cdot f(h(y))$, holds for each $f \in \mathcal{A}$ if and only if $y \in Y_c$.

(e) Y_c is closed in $h^{-1}(X)$.

(f) $h(Y_d)$ is a subset of the limit points of X_∞.

(g) The set $h(Y_d) \cap \mathrm{int}(K)$ is finite, for every compact subset K of X.

(h) If T is injective, then $h(Y_0)$ is a dense subset of X_∞.

Proof. See the proof of propositions 3,4 and 5 in [**6**].

□

Now we prove some results in the following two lemmas which will be used in section 4.

Lemma 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be regular Banach function algebras with structure spaces X and Y, respectively, where $\mathcal{A}$ satisfies the Ditkin's condition. Let T :*

$\mathcal{A}\longrightarrow\mathcal{B}$ be a disjointness preserving operator with finite nullity m. Then $X\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$ consists of k isolated points, where $k \le m$. Moreover, for each compact subset K of X we have, $\mathrm{int}(K)\cap \mathrm{cl}_{X_\infty}(h(Y_c)) = \mathrm{int}(K)\cap \mathrm{cl}_{X_\infty}(h(Y_0))$, consequently $X\backslash \mathrm{cl}_{X_\infty}(h(Y_c)) = X\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$.

Proof. Assume first that there exist distinct isolated points $x_1, x_2, ..., x_{m+1}$in $X\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$ and let $V_1, V_2, ..., V_{m+1}$ be disjoint open neighborhoods of elements $x_1, x_2, ..., x_{m+1}$ in $X\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$, respectively. For each $i = 1, 2, ..., m+1$ let U_i be an open neighborhood of x_i such that $\mathrm{cl}_{X_\infty}(U_i) \subseteq V_i$. Then by the regularity of $\mathcal{A}$ for each $i = 1, ..., m+1$ there exists an element $f_i \in \mathcal{A}$, such that $f_i(x_i) = 1$ and $f_i = 0$ on $X\backslash U_i$. Let $y \in Y_c \cup Y_d = Y_0$ be an arbitrary point. We can see that $h(y) \notin \mathrm{cl}_{X_\infty}(\mathrm{coz}(f_i))$, which implies that $y \notin \mathrm{coz}(Tf_i)$ according to Proposition 3.1(b), that is, $Tf_i(y) = 0$. Since $Tf_i = 0$ on $Y\backslash Y_0$, we conclude that $Tf_i = 0$, i.e. $f_i \in \ker(T)$. This implies that $\dim \ker(T) \ge m+1$, since f_i's are linearly independent. This contradiction shows that the open subset $X\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$ of X consists of at most m isolated points. Now suppose that K is a compact subset of X. Then $\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_c)) \subseteq (\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_0))) \cup (\mathrm{int}(K)\cap h(Y_d))$, which implies that $\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_c))$ is a finite open subset of X, since both $\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$ and $\mathrm{int}(K)\cap h(Y_d)$ are finite by the above argument and by Proposition 3.1(g). Therefore, $\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_c))$ consists of isolated points. Using this fact that $h(Y_d)$ is a subset of the limit points of X_∞ (see Proposition 3.1(f)), we conclude that $(\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_c))) \cap h(Y_d) = \emptyset$ and consequently $\mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_c)) \subseteq \mathrm{int}(K)\backslash \mathrm{cl}_{X_\infty}(h(Y_0))$. Therefore, $\mathrm{int}(K)\cap \mathrm{cl}_{X_\infty}(h(Y_c)) = \mathrm{int}(K)\cap \mathrm{cl}_{X_\infty}(h(Y_0))$, which implies that $X\cap \mathrm{cl}_{X_\infty}(h(Y_c)) = X\cap \mathrm{cl}_{X_\infty}(h(Y_0))$ and hence the final result follows. □

In the following definition we assume that $\mathcal{A}$ and $\mathcal{B}$ be regular Banach function algebras with structure spaces X and Y, respectively, where $\mathcal{A}$ satisfies the Ditkin's condition and $T : \mathcal{A}\longrightarrow\mathcal{B}$ be a disjointness preserving operator.

Definition 3.1. We define an equivalence relation $\sim$ on Y_c such that $y \sim y'$ if and only if $h(y) = h(y')$. For $y \in Y_c$ let $[y]$ be the equivalence class of y. We define $M = \{y \in Y_c : \mathrm{card}([y]) > 1\}$ and $m(T) = \mathrm{card}(\bigcup([y]\backslash\{y\})) = \sum\{\mathrm{card}([y]) - 1 : [y] \in Y_c/_\sim\}$, where the union is taken over all distinct elements $[y] \in Y_c/_\sim$ with $y \in M$.

Remark 3.3. *For each $f \in \mathcal{A}$, Proposition 3.1(d) shows that if $Tf(y) = 0$, for some $y \in Y_c$ then $Tf(y') = 0$ for all $y' \in [y]$.*

Lemma 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be regular Banach function algebras with structure spaces X and Y, respectively, where $\mathcal{A}$ satisfies the Ditkin's condition. Let $T : \mathcal{A}\longrightarrow\mathcal{B}$ be a disjointness preserving Fredholm operator with finite nullity m and corank n, then $m(T) + \mathrm{card}(Y\backslash Y_0) \le n$.*

Proof. Suppose that the inequality $m(T)+\mathrm{card}(Y\backslash Y_0)\leq n$ does not hold, i.e., there exist $y_{(1,0)},\ y_{(2,0)}\ ...,\ y_{(t_0,0)}\in Y\backslash Y_0$ and $x_1,...,x_k\in h(Y_c)$ with corresponding points $y_{(1,j)},y_{(2,j)},...,y_{(t_j,j)}\in h^{-1}(x_j)\cap Y_c$ for $j=1,2,...,k$ such that $\sum_{j=1}^{k}(t_j-1)+t_0\geq n+1$. For $j=0,1,2,...,k$, let $g_{(i,j)}\in\mathcal{B}$ such that $g_{(i,j)}(y_{(i,j)})=1$ and $g_{(i,j)}(y_{(i',j')})=0$, whenever $i\neq i'$ or $j\neq j'$, for $1\leq i\leq t_j-1$ and for $j=0$, $1\leq i\leq t_0$. We can assume that $g_{(i,j)}$'s have disjoint supports. Now consider the following subset of B, $G=\{g_{(1,0)},...,g_{(t_0,0)},g_{(1,1)},...,g_{(t_1-1,1)},...,g_{(1,k)},...,g_{(t_k-1,k)}\}$. Using Remark 3.3 we show that G has no intersection with $T(\mathcal{A})$. Indeed, for $1\leq i\leq t_0$ if $g_{(i,0)}=Tf$, for some $f\in\mathcal{A}$, then $g_{(i,0)}(y_{(i,0)})=Tf(y_{(i,0)})=0$, which is a contradiction. Now if there exists $f\in\mathcal{A}$ such that $g_{(i,j)}=Tf$, for some $j=1,2,...,k$ and $1\leq i\leq t_j-1$, then since for each $i'\neq i$, $1\leq i'\leq t_j-1$, $y_{(i,j)}$ and $y_{(i',j)}$ are in the same equivalence class and moreover, $g_{(i,j)}(y_{(i',j)})=0$, it follows that $g_{(i,j)}(y_{(i,j)})=0$, which is again a contradiction. We now show that the elements of G are linearly independent functions in $\mathcal{B}$ modulo the range of T. In fact, if $g=\sum\lambda_{(i,j)}g_{(i,j)}\in T(\mathcal{A})$, where the sum is taken over all (i,j) with $g_(i,j)\in G$, and $\lambda_{(i,j)}$ are complex numbers, then since $y_{(i,0)}\in Y\backslash Y_0$ for $1\leq i\leq t_0$, it follows that $g(y_{(i,0)})=0$, which implies easily that $\lambda_{(i,0)}=0$ for all $1\leq i\leq t_0$. On the other hand, since for each $j=1,2,...,k$, $g(y_{(t_j,j)})=0$, and $y_{(t_j,j)}$ is in the same equivalence class of $y_{(i,j)}$ for all $1\leq i\leq t_j-1$ it follows that $\lambda_{(i,j)}=g(y_{(i,j)})=0$. Therefore, $\dim\ (\mathcal{B}/_{T(\mathcal{A})})\geq n+1$, which is a contradiction. Hence $m(T)+\mathrm{card}(Y\backslash Y_0)\leq n$ as desired. □

4. Weighted composition Fredholm operators

In this section we give some results on weighted composition fredholm operator $T:\mathcal{A}\longrightarrow\mathcal{B}$, as an example of disjointness preserving fredholm linear maps, defined between certain regular Banach function algebras $\mathcal{A}$ and $\mathcal{B}$ by $Tf(y)=\omega(y)f(h(y))$, for $f\in\mathcal{A}$ and an appropriate function $h:Y\longrightarrow X$ and for a non-vanishing function $\omega:Y\longrightarrow\mathbb{C}$. In the sequel $\mathcal{A},\mathcal{B}$ are regular Banach function algebras with structure spaces X and Y, respectively, where $\mathcal{A}$ satisfies the Ditkin's condition and $\mathcal{B}$ is a uniform algebra.

Lemma 4.1. *Let $T:\mathcal{A}\longrightarrow\mathcal{B}$ be a weighted composition operator of the form $Tf(y)=\omega(y)f(h(y))$, $y\in Y$ and $f\in\mathcal{A}$, where $h:Y\longrightarrow X$ and $\omega:Y\longrightarrow\mathbb{C}$ are continuous functions and ω is non-vanishing. If T has a closed range, then there exists a positive constant r such that for each $x\in h(Y)$*

$$0<r\leq\sup_{y\in h^{-1}(\{x\})}|\omega(y)|.$$

Proof. We note that, using the closed graph theorem, T is continuous. First consider the case that T is injective. Then since T has a closed range it follows easily that T is bounded bellow. So there exists a positive real number r such that $r\|f\|_{\mathcal{A}} \le \|Tf\|_Y$ for all $f \in \mathcal{A}$. Let $x \in h(Y)$ and U, V be open neighborhoods of x in X such that $\mathrm{cl}_X(U)$ is compact and $\mathrm{cl}_X(U) \subseteq V$. By the regularity of $\mathcal{A}$ there exists a function $f_{UV} \in \mathcal{A}$ such that $f_{UV} = 1$ on U and $f_{UV} = 0$ on $X \backslash V$.Therefore

$$r\|f_{UV}\|_{\mathcal{A}} \le \|Tf_{UV}\|_Y = \sup_{y \in Y} |\omega(y) f_{UV}(h(y))| = \sup_{h(y) \in V} |\omega(y)||f_{UV}(h(y))|$$

$$\le \sup_{h(y) \in V} |\omega(y)|\|f_{UV}\|_X \le \sup_{h(y) \in V} |\omega(y)|\|f_{UV}\|_{\mathcal{A}}.$$

Therefore, $r \le \sup_{y \in h^{-1}(V)} |\omega(y)|$. The above argument shows that we can choose a net $\{y_\lambda\}$ in Y and take $\epsilon > 0$ small enough, such that $h(y_\lambda) \longrightarrow x$ and $|\omega(y_\lambda)| > r - \varepsilon$. Passing through a subnet we can assume that $y_\lambda \longrightarrow y_0$ for some $y_0 \in Y_\infty$. Hence for all U and V as above we have $|Tf_{UV}(y_\lambda)| = |\omega(y_\lambda)| > r - \varepsilon > 0$, for sufficiently large λ, thus $y_0 \ne \infty$ and hence $y_0 \in Y$. Therefore $h(y_0) = x$ and $r - \varepsilon \le |\omega(y_0)| \le \sup_{y \in h^{-1}(\{x\})} |\omega(y)|$. Since ε can be arbitrary small, this implies the desired inequality. Now assume that T is not injective. Hence in this case $h(Y)$ cannot be dense in X. Set $I = \ker(T)$, then clearly I is a closed ideal in $\mathcal{A}$ and so $\mathcal{A}/I$ is a Banach algebra with structure space $\sigma(\mathcal{A}/I) = h_{\mathcal{A}}(I)$. We first show that $h_{\mathcal{A}}(I) = \mathrm{cl}_X(h(Y))$. Obviously $\mathrm{cl}_X(h(Y)) \subseteq h_{\mathcal{A}}(I)$. Conversely, let $x_0 \in h_{\mathcal{A}}(I)$ and assume on the contrary that $x_0 \notin \mathrm{cl}_X(h(Y))$. Then by the regularity of $\mathcal{A}$, there exists a function $f \in \mathcal{A}$ such that $f(x_0) = 1$ and $f = 0$ on $h(Y)$. Therefore, $Tf(y) = \omega(y) \cdot f(h(y)) = 0$ for all $y \in Y$ which implies $f \in I$. On the other hand we have $f(x_0) \ne 0$ which is a contradiction. Thus $h_{\mathcal{A}}(I) = \mathrm{cl}_X(h(Y))$. We now show that $\mathcal{A}/I$ is semisimple. for each $y \in Y$ let $\varphi_{h(y)} : \mathcal{A}/I \longrightarrow \mathbb{C}$ be defined by $\varphi_{h(y)}(f + I) = f(h(y))$. Clearly $\varphi_{h(y)}$ is well-defined and is a non-zero complex homomorphism on $\mathcal{A}/I$. Hence if $f \in \mathcal{A}$ and $f + I \in \mathrm{Rad}(\mathcal{A}/I)$, then, $f(h(y)) = 0$ for each $y \in Y$, which implies that $Tf = 0$, that is $f \in I$. Therefore $\mathcal{A}/I$ is semisimple and we can consider $\mathcal{A}/I$ as a Banach function algebra on its maximal ideal space. Through this identification $\varphi_{h(y)}$ is, indeed, the same evaluation homomorphism $\delta_{h(y)}$. Now let $\widetilde{T} : \mathcal{A}/I \longrightarrow \mathcal{B}$ be defined with $\widetilde{T}(f + I) = Tf$ then $\widetilde{T}$ is injective and is a weighted composition operator of the form $\widetilde{T}(f + I)(y) = \omega(y) \cdot \delta_{h(y)}(f + I)$. Clearly $\widetilde{T}$ has a closed range as well. Therefore the conclusion follows from the first part of proof. □

Corollary 4.2. *Under the hypotheses of the above lemma, $h(Y)$ is a closed subset of X.*

Proof. Let $x_0 \in \mathrm{cl}_X(h(Y))$ and $x_0 \notin h(Y)$. Then there exists a net $\{y_\lambda\}$ in Y such that $h(y_\lambda) \ne x_0$ and $h(y_\lambda) \longrightarrow x_0$. Using the above lemma for each $h(y_\lambda)$ instead of x and replacing each y_λ by an appropriate point in Y we can assume that $\{\omega(y_\lambda)\}$ is away from zero. By passing through a subnet, if necessary , we can also assume that $y_\lambda \longrightarrow y_0$ for some $y_0 \in Y_\infty$. If $y_0 \in Y$, then $h(y_0) = x_0$ which is impossible, thus

$y_0 = \infty$. Therefore, $0 = \lim_\lambda Tf(y_\lambda) = \lim_\lambda \omega(y_\lambda) f(h(y_\lambda))$, for each $f \in \mathcal{A}$, which implies that $f(x_0) = \lim_\lambda f(h(y_\lambda)) = 0$, for each $f \in \mathcal{A}$, which is impossible. Thus $h(Y)$ is closed in X. □

Before stating the next theorem we note that for each isolated point $x \in X$, the regularity of $\mathcal{A}$ shows that the characteristic function $\chi_{\{x\}}$ is an element of $\mathcal{A}$.

Theorem 4.3. *Under the hypothesis of the above lemma if T is Fredholm with nullity m and corank n, then*

(a) $X \backslash h(Y) = \{x_1, ..., x_m\}$, where $x_1, ..., x_m$ are isolated points of X. Moreover, $\ker(T) = \operatorname{span}\{\chi_{\{x_1\}}, ..., \chi_{\{x_m\}}\}$.

(b) ω is away from zero, i.e. there exists a positive real number r such that for each $y \in Y$, $0 < r \leq |\omega(y)|$.

(c) $h : Y \backslash M \longrightarrow h(Y) \backslash h(M)$ and $\tilde{h} : Y/_\sim \longrightarrow h(Y)$, $[y] \mapsto h(y)$ are homeomorphism.

Proof. (a) We first note that since h and ω are continuous and $Tf(y) = \omega(y) f(h(y))$ for all $y \in Y$ it follows easily that $Y = Y_c = Y_0$ where Y_c and Y_0 are the subsets associated to the separating map T in Proposition 3.1. Using Lemma 3.2 we have $X \backslash h(Y) = \{x_1, ..., x_k\}$ where $k \leq m$ and $x_1, ..., x_k$ are isolated points of X. So it suffices to show that $k = m$ and $\{\chi_{\{x_1\}}, ..., \chi_{\{x_m\}}\}$ generates $\ker(T)$. It is clear that for a function $f \in \mathcal{A}$, $f \in \ker(T)$, if and only if $f = 0$ on $h(Y)$, if and only if there exist $\lambda_1, ..., \lambda_k$ such that $f = \sum_{i=1}^k \lambda_i \chi_{x_i}$. Hence $\ker(T) = \operatorname{span}$ $\{\chi_{\{x_1\}}, ..., \chi_{\{x_k\}}\}$. Since $\{\chi_{\{x_i\}}\}_{i=1}^k$ is linearly independent and $\dim \ker(T) = m$, we conclude that $k = m$, as desired.

(b) Using the same argument as in Lemma 3.4 we obtain $m(T) < \infty$, which implies, in particular, that M is a finite subset of Y. By Lemma 4.1, there exists $r_1 > 0$, such that $0 < r_1 \leq \sup_{y \in h^{-1}(\{x\})} |\omega(y)|$ for all $x \in h(Y)$. Then it is easy to see that for $r = \min\{r_1, |\omega(y)| : y \in M\}$, the inequality $0 < r \leq |\omega(y)|$ holds for all $y \in Y$.

(c) Obviously the restriction map $h : Y \backslash M \longrightarrow h(Y) \backslash h(M)$ is a bijective continuous map. We shall prove that the inverse map $h^{-1} : h(Y) \backslash h(M) \longrightarrow Y \backslash M$ is continuous as well. Let $\{h(y_\lambda)\}_\lambda$ be a net in $h(Y) \backslash h(M)$, such that $h(y_\lambda) \longrightarrow h(y)$ for some $y \in Y \backslash M$ and assume on the contrary that $\{y_\lambda\}$ does not converge to y. Passing through a subnet, we may assume that $y_\lambda \longrightarrow y_0$ for some $y_0 \in Y_\infty$ with $y_0 \neq y$. If $y_0 = \infty$ then $0 = Tf(y_0) = \lim_\lambda Tf(y_\lambda) = \lim_\lambda \omega(y_\lambda) f(h(y_\lambda))$. Using part (b) we conclude that $\lim_\lambda (h(y_\lambda)) = 0$, i.e. $f(h(y)) = 0$ for each $f \in \mathcal{A}$, hence $h(y) = \infty$, which is a contradiction. Thus $y_0 \neq \infty$, i.e. $y_0 \in \mathcal{Y}$ and consequently $h(y_\lambda) \longrightarrow h(y_0)$. Therefore, $h(y) = h(y_0)$, which concludes that $y = y_0$ a contradiction. To prove that $\widetilde{h} : Y/_\sim \longrightarrow h(Y)$ is a homeomorphism, We first note that $\widetilde{h}$ is a continuous bijection. Hence it suffices to show that $\widetilde{h}$ is an open map. For, suppose that $\widetilde{U}$ is an open subset of $Y/_\sim$ and let $U = \{y \in Y : [y] \in \widetilde{U}\}$. Then U is an open subset of Y and since $h : Y\backslash M \longrightarrow h(Y)\backslash h(M)$ is a homeomorphism, so it suffices to show that each point $x \in \widetilde{h}(\widetilde{U}) \cap h(M)$ is an interior point of $\widetilde{h}(\widetilde{U})$. Suppose on the contrary that there exists $x \in \widetilde{h}(\widetilde{U}) \cap h(M)$ and $\{x_\lambda\}_\lambda$ in $h(Y)\backslash \widetilde{h}(\widetilde{U})$ such that for each λ, $x_\lambda \notin h(M)$ and $x_\lambda \longrightarrow x$. Let $y_\lambda = h^{-1}(x_\lambda)$. As the equivalence classes $[y_\lambda] \notin \widetilde{U}$ imply $y_\lambda \notin U$ then there exists a subnet $\{y_{\lambda_\alpha}\}$ of $\{y_\lambda\}$ and $y_0 \in Y_\infty \backslash U$ such that $y_{\lambda_\alpha} \longrightarrow y_0$. Let $y_0 \in Y$ then from this fact that $y_0 \notin U$ we imply $[y_0] \notin \widetilde{U}$, on the other hand $\widetilde{h}([y_0]) = h(y_0) = \lim_\alpha h(y_{\lambda_\alpha}) = \lim_\alpha x_{\lambda_\alpha} = x \in \widetilde{h}(\widetilde{U})$ which is a contradiction. Thus $y_0 = \infty$ and consequently $0 = \lim_\alpha Tf(y_{\lambda_\alpha}) = \lim_\alpha \omega(y_{\lambda_\alpha}) f(h(y_{\lambda_\alpha}))$. Now by part (b) we conclude that $\lim_\alpha f(h(y_{\lambda_\alpha})) = 0$. Hence $f(x) = 0$ for all $f \in \mathcal{A}$ which is impossible. Therefore, x is an interior point of $\widetilde{h}(\widetilde{U})$ and this completes the proof. □

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