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# Weakly principally quasi-Baer rings and generalized triangular matrix rings

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Abstract. Generalized triangular matrix rings are ubiquitous in algebra and have relevant applications to analysis. A ring $R$ is called weakly principally quasi-Baer or simply (weakly p.q.-Baer) if the right annihilator of a principal right ideal is right $s$-unital by right semicentral idempotents, which implies that $R$ modulo the right annihilator of any principal right ideal is flat. In this paper, we characterize when a generalized triangular matrix ring is a weakly p.q.-Baer ring. Examples to illustrate and delimit the theory are provided.

## 1. Introduction

Throughout this paper rings are associative with a nonzero unity, modules are unital. Recall from [16] that $R$ is a *Baer* ring if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. The study of Baer rings has its roots in Operator Theory. In [26] Rickart studied $C^*$-algebras with the property that every right annihilator of any element is generated by a projection. Using Rickart's work, Kaplansky [15] defined an AW$^*$-algebra as a $C^*$-algebra with the stronger property that right annihilators of nonempty subsets are genenrated by a projection. In [16] Kaplansky introduced Baer rings to abstract various properties of AW$^*$-algebras, von Neumann algebras, and complete $*$-regular rings. Berberian continued the development of Baer rings in [1]. The class of Baer rings includes the von Neumann algebras (e.g., the algebra of all bounded operators on a Hilbert space), the commutative $C^*$-algebra $C(T)$ of continuous complex valued functions on a Stonian space $T$, and the regular rings whose lattice of principal right ideals is complete (e.g., regular rings which are continuous or right self-injective).

Closely related to Baer rings are principally projective (PP) rings. A ring $R$ is called a *right (resp. left) PP ring* if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of $R$ is

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generated (as a right (resp. left) ideal) by an idempotent of $R$). $R$ is called a PP ring if it is both right and left PP. The concept of PP ring is not left-right symmetric by Chase [**10**]. A right PP ring $R$ is Baer (so PP) when $R$ is orthogonally finite by Small [**27**] (where $R$ is *orthogonally finite* if has no infinite set of orthogonal idempotents).

A ring $R$ is called *quasi-Baer* if the right annihilator of every right ideal of $R$ is generated as a right ideal by an idempotent. It is easy to see that the quasi-Baer property is left-right symmetric for any ring. Quasi-Baer rings were initially considered by Clark [**11**] and used to characterize a finite dimensional algebra over an algebraically closed field as a twisted semigroup algebra of a matrix units semigroup. In [**25**], Pollingher and Zaks show that the class of quasi-Baer rings is closed under $n$-by-$n$ matrix rings and under $n$-by-$n$ upper (or lower) triangular matrix rings. Birkenmeier et al. [**7**] obtained a structure theorem, via triangulating idempotents, for an extensive class of quasi-Baer rings which includes all piecewise domains. Some results on quasi-Baer rings can be found in (cf. [**2**], [**5**], [**6**], [**7**], [**8**] and [**25**]).

Birkenmeier, Kim and Park in [**4**] introduced the concept of principally quasi-Baer rings. A ring $R$ is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, $R$ is right p.q.-Baer if $R$ modulo the right annihilator of any principal right ideal is projective. If $R$ is a semiprime ring, then $R$ is right p.q.-Baer if and only if $R$ is left p.q.-Baer. The class of right p.q.-Baer rings includes properly the class of quasi-Baer rings. Some examples were given in [**4**] to show that the classes of right p.q.-Baer rings and right PP rings are distinct.

Following Tominaga [**29**], an ideal $I$ of $R$ is said to be *left s-unital* if, for each $a \in I$, there is an element $x \in I$ such that $xa = a$. According to Liu and Zhao [**17**], a ring $R$ is called a *right APP-ring* if the right annihilator $r_R(aR)$ is left $s$-unital as an ideal of $R$ for any element $a \in R$ [**17**]. Left *APP*-rings may be defined analogously. This concept is a common generalization of left p.q.-Baer rings and right PP rings. In [**17**], authors showed that the APP property is inherited by polynomial extensions and is a Morita invariant property.

As a generalization of p.q.-Baer rings, Majidinya and Moussavi in [**18**] introduced the concept of *weakly p.q.-Baer* rings. A ring $R$ with unity is *weakly p.q.-Baer* if for each $a \in R$ there exists a nonempty subset $E$ of left semicentral idempotents of $R$ such that $r_R(aR) = \bigcup_{e\in E} eR$. The class of weakly p.q.-Baer rings is a natural subclass of the class of APP rings and includes both left p.q.-Baer rings and right p.q.-Baer rings. The properties of these rings have been investigated by many authors (see, [**17**], [**19**], [**20**], [**24**], and [**29**], for instance ).

Throughout this note, let $R$ and $S$ be rings with unity, $M$ a left $R$, right $S$ bimodule and

$$T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

be the generalized triangular matrix ring. Generalized triangular matrix rings have proven to be extremely useful in ring theory. They provide a good source of examples and counter examples as well as providing a framework to explore the connections between $End(M_R)$, $M$ and $R$ when $S = End(M_R)$. Generalized triangular rings have been the topic of a large number of publications, (see, for example [6], [12], [13], [14], [21], [22], [23] and [28]). They are useful in many areas of algebra (e.g., finite dimensional algebras and Morita contexts) and functional analysis (e.g., operator algebras see [9, p. 118-119]).

In this paper, we characterize when a generalized triangular matrix ring is weakly p.q.-Baer. To reach this aim, we identify the annihilators of ideals and the semicentral idempotents of a generalized triangular matrix ring. Then we provide a characterization of the generalized matrix rings which are Weakly principally quasi-Baer. As an application, we determine when the ring $T_n(R)$ of upper triangular matrices over $R$ is a weakly p.q.-Baer ring. Finally, we provide examples to illustrate these concepts.

Recall from [3] that an idempotent $e \in R$ is called *left (resp. right) semicentral* if $xe = exe$ (resp. $ex = exe$), for all $x \in R$. Equivalently, $e$ is left (respectively, right) semicentral if $eR$ (respectively, $Re$) is an ideal. If $e$ is both left and right semicentral, then it is central. The set of left (resp. right) semicentral idempotents of $R$ is denoted by $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$).

For a non-empty subset $X$ of $R$, $r_R(X)$ (resp. $\ell_R(X)$) is used for the right (resp. left) annihilator of $X$ over $R$. Furthermore, for a ring $R$, we denote by $Z_r(R)$, $M_n(R)$ and $T_n(R)$ the right singular ideals of $R$, the ring of $n \times n$ matrices over $R$ and the ring of $n \times n$ upper triangular matrices over $R$, respectively.

## 2. Characterizations of weakly principally quasi-Baer rings of generalized triangular matrix rings

This section is devoted to characterizations of generalized triangular matrix rings that are weakly principally quasi-Baer. Here we describe the ingredients of our main results, including the annihilators of ideals and the semicentral idempotents of a generalized triangular matrix ring.

The following lemma describes the annihilators of ideals of generalized triangular matrix rings.

**Lemma 2.1.** [6, Lemma 3.1] *Let* $T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a generalized triangular matrix ring and* $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ *a right ideal of* $T$. *Then:*

$$r_T \begin{pmatrix} I & N \\ 0 & J \end{pmatrix} = \begin{pmatrix} r_R(I) & r_M(I) \\ 0 & r_S(J) \cap r_S(N) \end{pmatrix}.$$

To characterize the generalized triangular matrix rings that are a weakly p.q.-Baer ring, we will need the following result. This lemma plays a fundamental role to achieve our aim in Theorem 2.4.

**Lemma 2.2.** [8, Lemma 2.5] *Let* $d = \begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ *be a left semicentral idempotent in* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. *Then* $dT = cT$ *where* $c = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$.

**Remark 2.3.** [18, Remark 2.15] *If the ideals* $I$ *and* $J$ *are left s-unital by left semicentral idempotents, then so is* $I \cap J$.

Now we are ready to characterize generalized upper triangular matrix rings that are weakly p.q.-Baer.

**Theorem 2.4.** *Let* $T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a generalized triangular matrix ring. Then the following are equivalent:*

(1) $T$ *is a weakly p.q.-Baer ring.*

(2) (*i*) $R$ *and* $S$ *are weakly p.q.-Baer;*

(*ii*) *for each* $r \in R$, $x \in r_R(rR)$ *and* $y \in r_M(rR)$ *there exists a left semicentral idempotent* $e \in r_R(rR)$ *such that* $x = ex$ *and* $y = ey$;

(*iii*) *for each* $r \in R$ *and* $m \in M$, *the ideal* $r_S(rM + mS)$ *is left s-unital by left semicentral idempotent.*

Proof. (1) $\Rightarrow$ (2). Assume that $r \in R$, $s \in S$ and $m \in M$. By Lemma 2.1, we have $r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} T\right) = \begin{pmatrix} r_R(rR) & r_M(rR) \\ 0 & r_S(sS) \cap r_S(rM + mS) \end{pmatrix}$. Since $T$ is a weakly principally quasi-Baer ring, $r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} T\right)$ is left $s$-unital by left semicentral. Then for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} T\right)$, there exists $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ in $\mathcal{S}_\ell(T) \cap r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} T\right)$ such that $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$. From

Lemma 2.2 it follows that there exists an left semicentral idempotent $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ in $r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}T\right)$ such that $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$.

$(i)$ Since for each $r \in R$, $r_T\left(\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}T\right)$ is left $s$-unital by left semicentral, $r_R(rR)$ is left $s$-unital by left semicentral. Also, for each $s \in S$, $r_T\left(\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}T\right)$ is left $s$-unital by left semicentral, $r_S(sS)$ is left $s$-unital by left semicentral. Therefore, $R$ and $S$ are weakly principally quasi-Baer.

$(ii)$ It is clear.

$(iii)$ Since for each $r \in R$ and $m \in M$, $r_T\left(\begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}T\right)$ is left $s$-unital by left semicentral, $r_S(rM + mS)$ is left $s$-unital by left semicentral.

(2) $\Rightarrow$ (1). Since $r_S(rM + mS)$ and $r_S(sS)$ are left $s$-unital by left semicentral, $r_S(rM+mS) \cap r_S(sS)$ is also left $s$-unital by left semicentral. Therefore, for any $z \in r_S(rM+mS) \cap r_S(sS)$ there exists an left semicentral idempotent $f \in S$ such that $z = fz$. From the condition $(iii)$ we infer that for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_T\left(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}T\right)$, there exists $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in \mathcal{S}_\ell(T)$ such that $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$. So $T$ is a weakly principally quasi-Baer ring which completes the proof. $\square$

The following result was proven in [18, Theorem 2.20]. But as an application of Theorem 2.4, we give a direct and different proof.

**Theorem 2.5.** [18, Theorem 2.20] *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is weakly p.q.-Baer;*
(2) *$T_n(R)$ is a weakly p.q.-Baer ring for every positive integer $n$;*
(3) *$T_n(R)$ is a weakly p.q.-Baer ring for some positive integer $n$.*

Proof. It is clear if $n = 1$. If $n > 1$ then $T_{n+1}(R) = \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$, where $M = (R, R, \dots, R)$ ($n$-tuple). Now, the result follows by induction on $n$ using Theorem 2.4. $\square$

The following corollary is an immediate consequence of Theorem 2.4 and provides a rich source of rings which are weakly p.q.-Baer.

**Corollary 2.6.** *Let $R$ be a weakly p.q.-Baer ring and $S$ a unitary subring of $R$ such that $r_R(aS) = 0$ for any $0 \neq a \in S$. Then the ring $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$ is a weakly p.q.-Baer ring.*

**Example 2.7.** *Let $D$ be a domain, $R = M_2(D)$ and* $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} a \in D \right\}$. *Since $D$ is a domain, $R$ is weakly p.q.-Baer. For any $0 \neq a \in S$, $r_R(aS) = 0$. Then by Corollary 2.6,* $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$ *is a weakly p.q.-Baer ring.*

As an application of Corollary 2.6, we obtain a class of examples of weakly p.q.-Baer rings that are neither PP and nor nonsingular.

**Example 2.8.** *Let* $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$, *where $R$ is a prime and $S$ is a unitary subring of the center of $R$. Then from Corollary 2.6 it follows that $T$ is a weakly p.q.-Baer ring. Moreover, if $Z_r(R) \neq 0$, then $Z_r(T) \neq 0$. Since every right PP ring is right nonsingular, it implies that $T$ is not right PP ring.*

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