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Stability of quartic functional equation in paranormed spaces

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Abstract. In this paper, we prove the Ulam-Hyers stability of the following quartic functional equation

$$f\left(\frac{x+y-2z}{2}\right)+f\left(\frac{y+z-2x}{2}\right)+f\left(\frac{z+x-2y}{2}\right) = \frac{9}{16}\left(f(x-y)+f(y-z)+f(z-x)\right)$$

in paranormed spaces using both direct and fixed point methods.

1. Introduction

The concept of the stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, the first stability problem concerning group homomorphisms was raised by Ulam [31] and affirmatively solved by Hyers [10] in 1941. Further the result of Hyers was generalized by Aoki [1] in 1950 for approximate additive mappings and by Rassias [27] for approximate linear mappings by allowing the normed difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\epsilon\left(||x||^p+||y||^p\right)$. Rassias [25] followed the innovative approach of Rassias Theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\,\|y\|^q$ with $p+q\neq 1$. In 1994, a generalization of Rassias Theorem was obtained by Găvruţa [9], who replaced $\epsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$. In 2008, a special case of Găvruţa's Theorem for the unbounded Cauchy difference was obtained by Ravi *et al.* [28] by considering the summation of both the sum and the product of two p-norms in the spirit of Rassias approach.

The quartic functional equation was first introduced by Rassias [26], who solved its Ulam stability problem. Later, Lee et al. [14] remodified Rassias' quartic

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functional equation and obtained its general solution. Park [23] proved the stability of quartic functional equation in orthogonality spaces.

The stability problems of various quartic functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces, orthogonal spaces and many other spaces have been broadly investigated by a number of mathematicians (see [3, 4, 12, 15, 17, 20, 21, 22]).

Recently, Vijayakumar, Karthikeyan, Rassias and Baskaran [32] discussed the solution in vector spaces and proved the Ulam-Hyers stability of the quartic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right) = \frac{9}{16}\left(f(x-y)+f(y-z)+f(z-x)\right) \tag{1}$$

originating from the sum of the medians of a triangle in fuzzy normed space by using both direct and fixed point methods.

In this paper, we prove the Ulam-Hyers stability of the quartic functional equation (1) in paranormed spaces using both direct and fixed point methods.

2. Basic concepts on paranormed spaces

The concept of statistical convergence for sequences of real numbers was introduced by Fast [7] and Steinhaus [30] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [8, 11, 18, 19, 29]). This notion was defined in normed spaces by Kolk [13].

We recall some basic facts concerning Fréchet spaces.

Definition 2.1. [33] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(P4) if $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n\, x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

Definition 2.2. [33] The paranorm is called total if, in addition, we have

(P5) $P(x) = 0$ implies $x = 0$.

Definition 2.3. [33] A Fréchet space is a total and complete paranormed space.

3. Stability results: Hyers' direct method

In this section, we investigate the Ulam-Hyers stability of the functional equation (1) in paranormed spaces using direct method.

Throughout this section, let (U, P) be a Fréchet space and $(V, \|\cdot\|)$ be a Banach space.

For the convenience, we define a mapping $F : U^3 \to V$ by

$$\begin{aligned} F(x,y,z) =& f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right) \\ &-\frac{9}{16}\left(f(x-y)+f(y-z)+f(z-x)\right) \end{aligned}$$

for all $x, y, z \in U$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed and $\alpha : U^3 \to [0,\infty)$ be a function with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{4nj}} \alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right) = 0 \tag{2}$$

for all $x, y, z \in U$. Suppose that an even mapping $f : U^3 \to V$ satisfies the following inequality

$$P\left(F(x,y,z)\right) \le \alpha(x,y,z) \tag{3}$$

for all $x, y, z \in U$. Then there is a unique quartic mapping $\mathcal{Q} : U^3 \to V$ such that

$$P\left(f(x)-\mathcal{Q}(x)\right) \le \sum_{k=\frac{1-j}{2}}^{\infty} \frac{1}{2^{4(k+1)j}} \alpha\left(2^{(k+1)j}x, 2^{(k+1)j}x, 0\right) \tag{4}$$

for all $x \in U$. The mapping $\mathcal{Q}(x)$ is defined by

$$P\left(\lim_{n\to\infty} \frac{f\left(2^{nj}x\right)}{2^{4nj}} - Q(x)\right) \to 0 \tag{5}$$

for all $x \in U$.

Proof. Assume that $j = 1$. Replacing (x, y, z) by $(x, x, 0)$ in (3), we get

$$P\left(2^4 f\left(\frac{x}{2}\right) - f(x)\right) \le \alpha(x,x,0) \tag{6}$$

for all $x \in U$ and all $r > 0$. Replacing x by $2x$ and divided by 2^4 in (6), we obtain

$$P\left(f\left(x\right) - \frac{f(2x)}{2^4}\right) \le \frac{1}{2^4}\alpha(2x,2x,0) \tag{7}$$

for all $x \in U$ and all $r > 0$. Replacing x by $2x$ and divided by 2^4 in (7), we obtain

$$P\left(\frac{f\left(2x\right)}{2^4} - \frac{f\left(4x\right)}{2^8}\right) \le \frac{1}{2^8}\alpha\left(4x,4x,0\right) \tag{8}$$

for all $x \in U$. By (7) and (8), we obtain

$$P\left(f(x)-\frac{f(4x)}{2^8}\right) \leq \left(P\left(f(x)-\frac{f(2x)}{2^4}\right)+P\left(\frac{f(2x)}{2^4}-\frac{f(4x)}{2^8}\right)\right)$$
$$\leq \frac{1}{2^4}\alpha(2x,2x,0)+\frac{1}{2^8}\alpha(4x,4x,0) \tag{9}$$

for all $x \in U$. Using induction on positive integers n, we get

$$P\left(f(x)-\frac{f(2^n x)}{2^{4n}}\right) \leq \sum_{k=0}^{n-1}\frac{1}{2^{4(k+1)}}\alpha\left(2^{k+1}x,2^{k+1}x,0\right) \tag{10}$$
$$\leq \sum_{k=0}^{\infty}\frac{1}{2^{4(k+1)}}\alpha\left(2^{k+1}x,2^{k+1}x,0\right)$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$, replacing x by $2^m x$ and divided by 2^{4m} in (10), for any $m, n > 0$, we simplify

$$P\left(\frac{f(2^m x)}{2^{4m}}-\frac{f(2^{m+n}x)}{2^{4(m+n)}}\right)=\frac{1}{2^{4m}}P\left(f(2^m x)-\frac{f(2^{m+n}x)}{2^{4n}}\right)$$
$$\leq \sum_{k=0}^{\infty}\frac{1}{2^{4(k+m+1)}}\alpha\left(2^{k+m+1}x,2^{k+m+1}x,0\right)$$

for all $x \in U$ and all $m, n \geq 0$. This shows that the sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ is a Cauchy sequence. Since V is complete, there exists a mapping $\mathcal{Q} : U \to V$ by

$$P\left(\lim_{n\to\infty}\frac{f(2^n x)}{2^{4n}}-\mathcal{Q}(x)\right)\to 0,$$

for all $x \in U$. By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty}\frac{t_n f(2^n x)}{2^{4n}}-t\mathcal{Q}(x)\right)\to 0.$$

Letting $n \to \infty$ in (10), we see that (4) holds, for all $x \in U$. To show that $\mathcal{Q}$ satisfies (1), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ and divided by 2^{4n} in (3), we get

$$\frac{1}{2^{4n}}P\left(F\left(2^n x,2^n y,2^n z\right)\right)\leq \frac{1}{2^{4n}}\alpha\left(2^n x,2^n y,2^n z\right),$$

for all $x, y, z \in U$. Letting $n \to \infty$ in the above inequality and using the definition of $\mathcal{Q}(x)$, we see that

$$P\left(\mathcal{Q}\left(\frac{x+y-2z}{2}\right)+\mathcal{Q}\left(\frac{y+z-2x}{2}\right)+\mathcal{Q}\left(\frac{z+x-2y}{2}\right)\right.$$
$$\left.-\frac{9}{16}\left(\mathcal{Q}(x-y)+\mathcal{Q}(y-z)+\mathcal{Q}(z-x)\right)\right)=0, \tag{11}$$

for all $x, y, z \in U$. Using condition $P(5)$ in (11), we get

$$\mathcal{Q}\left(\frac{x+y-2z}{2}\right)+\mathcal{Q}\left(\frac{y+z-2x}{2}\right)+\mathcal{Q}\left(\frac{z+x-2y}{2}\right) \\ =\frac{9}{16}\left(\mathcal{Q}(x-y)+\mathcal{Q}(y-z)+\mathcal{Q}(z-x)\right)$$

for all $x, y, z \in U$. Hence $\mathcal{Q}$ satisfies (1), for all $x, y, z \in U$. In order to prove that $\mathcal{Q}(x)$ is unique, let $\mathcal{Q}'(x)$ be another quartic functional equation satisfying (1) and (4). Then

$$\begin{aligned} P(\mathcal{Q}(x)-\mathcal{Q}'(x)) &= \frac{1}{2^{4n}}\left\{P\left(\mathcal{Q}\left(2^n x\right)-\mathcal{Q}'\left(2^n x\right)\right)\right\} \\ &\le \frac{1}{2^{4n}}\left\{P\left(\mathcal{Q}\left(2^n x\right)-f\left(2^n x\right)\right)+P\left(f\left(2^n x\right)-\mathcal{Q}'\left(2^n x\right)\right)\right\} \\ &\le \sum_{k=0}^{\infty}\frac{1}{2^{4(k+n+1)}}\alpha\left(2^{k+n+1}x, 2^{k+n+1}x, 0\right) \to 0 \ as\ n\to\infty \end{aligned}$$

for all $x \in U$. Thus $P(\mathcal{Q}(x)-\mathcal{Q}'(x))=0$, for all $x \in U$. Hence we have $\mathcal{Q}(x)=\mathcal{Q}'(x)$. So $\mathcal{Q}(x)$ is unique. Thus the mapping $\mathcal{Q}: U^3 \to V$ is a unique quartic mapping.

For $j=-1$, we can prove the result by a similar method. This completes the proof. □

From Theorem 3.1, we obtain the following corollary concerning the Ulam-Hyers stability for the functional equation (1).

Corollary 3.2. *Let $F: U^3 \to V$ be a mapping and assume that there exist real numbers λ and s such that*

$$\begin{aligned} &P\left(F(x,y,z)\right) \\ &\le \begin{cases} \lambda, & \\ \lambda\left\{P(x)^s+P(y)^s+P(z)^s\right\}, & s\neq 4 \\ \lambda\left\{P(x)^sP(y)^sP(z)^s+\left\{P(x)^{3s}+P(y)^{3s}+P(w)^{3s}\right\}\right\}, & s\neq \frac{4}{3} \end{cases} \end{aligned} \tag{12}$$

for all $x, y, z \in U$. Then there exists a unique quartic mapping $\mathcal{Q}: U^3 \to V$ such that

$$P\left(f(x)-\mathcal{Q}(x)\right) \le \begin{cases} \dfrac{\lambda}{|15|} \\ \dfrac{2\lambda P(x)^s}{|2^4-2^s|} \\ \dfrac{2\lambda P(x)^{3s}}{|2^4-2^{3s}|} \end{cases} \tag{13}$$

for all $x \in U$.

4. Alternative stability results: Fixed point method

In this section, we prove the Ulam-Hyers stability of the functional equation (1) in paranormed spaces by using the fixed point method.

Now, we will recall the fundamental results in fixed point theory.

Theorem 4.1. [16] *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$$

or

$(B2)$ *there exists a natural number n_0 such that:*
(i) *$d(T^n x, T^{n+1}x) < \infty$, for all $n \geq n_0$;*
(ii) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*
(iv) *$d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$, for all $y \in Y$.*

Many researchers have applied the fixed point alternative method to prove the Ulam-Hyers stability of functional equations (see [2, 5, 6, 24]).

Theorem 4.2. *Let $F : U^3 \to V$ be a mapping for which there exists a function $\alpha : U^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{4n}} \alpha\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right) = 0 \tag{14}$$

where $\mu_0 = \frac{1}{2}$ and $\mu_1 = 2$ such that the functional inequality

$$P\left(F(x, y, z)\right) \leq \alpha(x, y, z) \tag{15}$$

for all $x, y, z \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \alpha\left(2x, 2x, 0\right)$$

has the property

$$\gamma(x) \leq L\ \frac{\gamma\left(\mu_i x\right)}{\mu_i^4}, \tag{16}$$

then there exists a unique quartic mapping $\mathcal{Q} : U^3 \to V$ satisfying (1) and

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{17}$$

for all $x \in U$.

Proof. Consider the set $\Omega = \{p/p : U^3 \to V,\ p(0, 0) = 0\}$ and introduce the generalized metric on Ω, $d(p, q) = d_\gamma(p, q) = \inf\{K \in (0, \infty) : P\left(p(x) - q(x)\right) \leq K\gamma(x), x \in U\}$. It is easy to see that (Ω, d) is complete.

Define $T : \Omega^3 \to \Omega$ by $Tp(x,x) = \frac{1}{\mu_i^4} p(\mu_i x)$, for all $x \in U$. Now $p, q \in \Omega$ imply $d(Tp, Tq) \leq L d(p,q)$, for all $p, q \in \Omega$, i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing (x,y,z) by $(x,x,0)$ in (15), implies that

$$P\left(2^4 f\left(\frac{x}{2}\right) - f(x)\right) \leq \alpha(x,x,0) \tag{18}$$

for all $x \in U$. By using (16) for the case $i = 1$, it reduces to

$$P\left(2^4 f\left(\frac{x}{2}\right) - f(x)\right) \leq \ \alpha(x,x,0) \leq \gamma(x)$$

for all $x \in U$, i.e.,

$$d_\gamma(Tf, f) \leq 1 \Rightarrow d(Tf, f) \leq L^0 < \infty.$$

Again replacing $x = 2x$ in (18), implies that

$$P\left(2^4 f(x) - f\left(\frac{x}{2}\right)\right) \leq \alpha(2x, 2x, 0). \tag{19}$$

By using (16) for the case $i = 0$, it reduces to

$$P\left(f(x) - \frac{1}{2^4} f\left(\frac{x}{2}\right)\right) \leq \ \frac{1}{2^4}\alpha(2x,2x,0) \leq L\gamma(x),$$

for all $x \in U$, i.e.,

$$d_\gamma(f, Tf) \leq L \Rightarrow d(f, Tf) \leq L^1 < \infty.$$

In both cases, we obtain that $d(g, Tg) \leq L^{1-i}$. Therefore $(A1)$ holds. From $(A2)$, it follows that there exists a fixed point $\mathcal{Q}$ of T in Ω such that

$$P\left(\lim_{n\to\infty} \frac{1}{\mu_i^{4n}} \left(f\left(\mu_i^{n+1}x\right) - f\left(\mu_i^n x\right)\right) - \mathcal{Q}(x)\right) \to 0, \tag{20}$$

for all $x \in U$. To prove $\mathcal{Q} : U^3 \to V$ is quartic. Replacing (x,y,z) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z)$ in (15) and divided by μ_i^{4n}, it follows from (14) that

$$P(\mathcal{Q}(x,y,z)) = \lim_{n\to\infty} P\left(\frac{1}{\mu_i^{4n}} F(\mu_i^n x, \mu_i^n y, \mu_i^n z)\right) \leq \lim_{n\to\infty} \frac{1}{\mu_i^{4n}} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0,$$

for all $x, y, z \in U$, i.e., $\mathcal{Q}$ satisfies the functional equation (1).

By $(A3)$, $\mathcal{Q}$ is the unique fixed point of T in the set $\Delta = \{\mathcal{Q} \in \Omega : d(f, \mathcal{Q}) < \infty\}$, $\mathcal{Q}$ is the unique mapping such that $P(f(x) - \mathcal{Q}(x)) \leq K\gamma(x)$, for all $x \in U$ and $K > 0$. Finally by $(A4)$, we obtain $d(f, \mathcal{Q}) \leq \frac{1}{1-L} d(f, Tf)$ which implies $d(f, \mathcal{Q}) \leq \frac{L^{1-i}}{1-L}$ which yields $P(f(x) - \mathcal{Q}(x)) \leq \frac{L^{1-i}}{1-L}\gamma(x)$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1).

Corollary 4.3. *Let $F : U^3 \to V$ be a mapping and there are real numbers λ and s such that the inequality (12) for all $x, y, z \in U$, then there is a unique quartic function $\mathcal{Q} : U^3 \to V$ such that the inequality (13) holds, for all $x \in U$.*

Proof. Set

$$\alpha(x,y,z) = \begin{cases} \lambda \\ \lambda\left\{P(x)^s + P(y)^s + P(z)^s\right\} \\ \lambda\left\{P(x)^s P(y)^s P(z)^s + \left\{P(x)^{3s} + P(y)^{3s} + P(z)^{3s}\right\}\right\} \end{cases}$$

for all $x, y, z \in U$. Now

$$\frac{1}{\mu_i^{4n}}\alpha\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right) = \begin{cases} \dfrac{\lambda}{\mu_i^{4n}}, \\ \dfrac{\lambda}{\mu_i^{4n}}\left\{P\left(\mu_i^n x\right)^s + P\left(\mu_i^n y\right)^s + P\left(\mu_i^n z\right)^s\right\} \\ \dfrac{\lambda}{\mu_i^{4n}}\left\{P\left(\mu_i^n x\right)^s P\left(\mu_i^n y\right)^s P\left(\mu_i^n z\right)^s + P\left(\mu_i^n x\right)^{3s} + P\left(\mu_i^n y\right)^{3s}\right. \\ \qquad \left. + P\left(\mu_i^n z\right)^{3s}\right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } n \to \infty \\ \to 0 \text{ as } n \to \infty \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus (14) holds. But we have $\gamma(x) = \alpha(2x, 2x, 0)$ has the property $\gamma(x) \leq L \cdot \frac{1}{\mu_i^4}\,\gamma\left(\mu_i x\right)$, for all $x \in U$. Hence

$$\gamma(x) = \alpha\left(2x, 2x, 0\right) = \begin{cases} \lambda, \\ 2\lambda 2^s P(x)^s, \\ 2\lambda 2^{3s} P(x)^{3s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^4}\gamma(\mu_i x) = \begin{cases} \mu_i^{-4}\lambda \\ 2\mu_i^{(s-4)}\lambda P(x)^s \\ 2\mu_i^{(3s-4)}\lambda P(x)^{3s} \end{cases} = \begin{cases} \mu_i^{-4}\gamma(x) \\ \dfrac{\mu_i^{s-4}}{2^s}\gamma(x) \\ \dfrac{\mu_i^{3s-4}}{2^{3s}}\gamma(x) \end{cases}$$

for all $x \in U$. Hence, the inequality (16) holds. Now from (17), we prove the following cases.

Case 1: $L = 2^{-4}$ for $s = 0$ if $i = 0$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{\lambda}{15}\right).$$

Case 2: $L = 2^4$ for $s = 0$ if $i = 1$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{1}{1-2^4}\gamma(x)\right) \leq P\left(\frac{\lambda}{-15}\right).$$

Case 3: $L = 2^{s-4}$ for $s < 4$ if $i = 0$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{\left(2^{s-4}\right)^{1-0}}{1-2^{s-4}}\right)\gamma(x) \leq \frac{2\lambda P(x)^s}{2^4 - 2^s}.$$

Case 4: $L = 2^{4-s}$ for $s > 4$ if $i = 1$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{\left(2^{4-s}\right)^{0}}{1-2^{4-s}}\right)\gamma(x) \leq \frac{2\lambda P(x)^s}{2^s - 2^4}.$$

Case 5: $L = 2^{3s-4}$ for $s < \frac{4}{3}$ if $i = 0$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{\left(2^{3s-4}\right)^{1-0}}{1-2^{3s-4}}\right)\gamma(x) \leq \frac{2\lambda P(x)^{3s}}{2^4 - 2^{3s}}.$$

Case 6: $L = 2^{4-3s}$ for $s > \frac{4}{3}$ if $i = 1$

$$P\left(f(x) - \mathcal{Q}(x)\right) \leq P\left(\frac{\left(2^{4-3s}\right)^{0}}{1-2^{4-3s}}\right)\gamma(x) \leq \frac{2\lambda P(x)^{3s}}{2^{3s} - 2^4}.$$

□

5. Conclusion

In this article, we have obtained the stability results and alternative stability results of quartic functional equation in paranormed spaces. Also when considering the stability results of the previously published paper [32] and the stability results of this paper, the mixed stability results are zero by the variables used here.

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