Mathematical Analysis and its Contemporary Applications
Volume 2, Issue 1, 2020, 35-42
doi: 10.30495/MACA.2020.679857

A new notion of affine sets

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Abstract. In this paper, we investigate the behaviour of e-convex sets and e-affine sets. Moreover, some notions like $S(e, a, \rho, \alpha)$ and e-affine cones are introduced and discussed. We complete with a role of above sets in linear idempotent maps.

1. Introduction

Geometrically, surfaces and curves generally reviewed to be set of points with outstanding features. Affine and convex spaces give an important framework for doing geometry. Any line is affine but a line segment is convex, but not affine (unless singleton). Hyperplanes are affine and convex but halfspaces are convex only. These inducements to concentrates on e-affine spaces. Youness [3] initiate e-convex sets and study optimality for some non-linear programming problems. Some notable result related to this notion are given in [1, 4, 5, 7, 8]. As a generalization of e-convex spaces, we introduce e-affine spaces with more counter examples. The framework of this paper as: for the first section, we list out the needed definitions and preliminary results. In section 2, we give out properties which related to e-convex sets. Section three is devoted to study the role of e-affine sets in idempotent injective maps. We have also given the idea of e-affine cone in section 4.

2. Notations and Preliminaries

Throughout this paper and for simplicity in appearance, e is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. We now recall some preliminaries from [2, 6].

Definition 2.1. *Any point s is of the form $s = \theta s_1 + (1 - \theta)s_2$, where $\theta \in \mathbb{R}$ $(0 \leq \theta \leq 1)$ is the affine combination (convex combination) of s_1 and s_2. A line (line segment) through s_1 and s_2 is described as all affine combinations (convex combinations) of s_1 and s_2.*

2010 *Mathematics Subject Classification.* 46N10, 47N10, 90C48, 90C90, 49K27.
Key words and phrases. Affine sets, e-convex sets, e-cone, e-affine set.

Definition 2.2. *Affine set (convex set) is a set that is closed under affine combinations (convex combinations), i.e., contains the line (line segment) through any two distinct points in the set.*

Example 2.3. *Solution set of linear equations* $\{s : As = b\}$ *is an example of a affine set.*

Lemma 2.4. *If a set* $S \subseteq \mathbb{R}^n$ *is e-convex, then* $e(S) \subseteq S$*, where* $e : \mathbb{R}^n \mapsto \mathbb{R}^n$*.*

Definition 2.5. *Let* $\alpha \in \mathbb{R}$*,* $a \in \mathbb{R}^n$ *and* $e : \mathbb{R}^n \mapsto \mathbb{R}^n$ *be any map.* $S(e, a, \rho, \alpha) = \{s \in \mathbb{R}^n :< e(s), a > \rho\alpha\}$ *where* $\rho \in \{<, =, >, \leq, \geq\}$ *and* $< ., . >$ *represents the inner product on* $\mathbb{R}^n$*. Then we have the following special case for S:*

i. $S(e, a, \geq, \alpha) = \{s \in \mathbb{R}^n :< e(s), a >\geq \alpha\}$ *is defined as the e-hyperplane,*
ii. $S(e, a, \leq, \alpha) = \{s \in \mathbb{R}^n :< e(s), a >\leq \alpha\}$ *is defined as the closed e-half space, and*
iii. $S(e, a, <, \alpha) = \{s \in \mathbb{R}^n :< e(s), a >< \alpha\}$ *is defined as the open e-half space.*

Definition 2.6. *A function* $e : \mathbb{R}^n \mapsto \mathbb{R}^n$ *is said to be invariant at* $a \in \mathbb{R}^n$ *if e(x+a)=e(x)+a where* $x \in \mathbb{R}^n$*.*

3. e-Affine Sets

In this section, we introduce e-affine sets and brood over some of their properties and investigate their relationship with convex sets.

Definition 3.1. *An e-affine set is a set* $S \subseteq \mathbb{R}^n$ *such that* $\theta e(s_1)+(1-\theta)e(s_2) \in S$ *for* $s_1, s_2 \in S$*,* $\theta \in \mathbb{R}$ *and e is a mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*.*

Proposition 3.2. *Every e-affine set is e-convex.*

Proof. Let $S \subseteq \mathbb{R}^n$ be e-affine. For every $s_1, s_2 \in S$ and $\theta \in \mathbb{R}$ we have $\theta e(s_1) + (1 - \theta)e(s_2) \in S$. Also this is holds for a particular θ such that $0 \leq \theta \leq 1$. Therefore S is e-convex. □

But an e-convex set need not be e-affine, the following example illustrates that.

Example 3.3. *Define* $e : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by* $e(s_1, s_2) = (0, s_2)$*. Let*

$$S_1 = \{(s_1, s_2) :\in \mathbb{R}^2 : (s_1, s_2) = \theta_1(0, 0) + \theta_1(2, 1) + \theta_3(0, 3)\}$$

and

$$S_2 = \{(s_1, s_2) :\in \mathbb{R}^2 : (s_1, s_2) = \theta_1(0, 0) + \theta_2(0, -3) + \theta_3(-2, -1)\}$$

where $\theta_1, \theta_2, \theta_3 \geq 0$*,* $S = S_1 \cup S_2$ *and* $\theta_1 + \theta_2 + \theta_3 = 1$*.*

For $(2,1),(0,3) \in S_1$. *Take* $\theta = -1$. *Then*

$$\begin{aligned}\theta\, e(s_1) + (1-\theta)\, e(s_2) &= -1\, e(2,1) + 2\, e(0,3)\\ &= (-1)\,(0,1) + 2\,(0,3)\\ &= (0,-1) + (0,6)\\ &= (0,5) \notin S_1.\end{aligned}$$

Therefore S *is not* e*-affine.*

Proposition 3.4. *If a set* $S \subseteq \mathbb{R}^n$ *is* e*-affine, then* $e(S) \subseteq S$.

Proof. Proof follows from Proposition 3.2 and Lemma 2.4. $\square$

The below example explains that there is an affine set but not e-affine for a particular operator e.

Example 3.5. *The set* $S = \{(s, s+1) : s \in \mathbb{R}\}$ *is affine. Let* $e : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $e(s_1, s_2) = (-s_1, -s_2)$. *Take* $(-1,0),(0,1) \in S$ *and* $\theta = 2$. *Then*

$$\begin{aligned}\theta\, e(-1,0) + (1-\theta)\, e(0,1) &= 2\,(1,0) + (-1)\,(0,-1)\\ &= (2,0) + (0,1) = (2,1) \notin S.\end{aligned}$$

Therefore S *is not* e*-affine.*

Proposition 3.6. *Let* $e : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a linear map such that* $e(S) \subseteq S$, *where* S *is a subspace of* $\mathbb{R}^n$. *Then* S *is* e*-affine.*

Proof. Subspace that S of $\mathbb{R}^n$ implies $\theta_1\, s_1 + \theta_2\, s_2 \in S$, for every $s_1, s_2 \in S$ and $\theta_1, \theta_2 \in \mathbb{R}$. By putting $\theta_2 = 1 - \theta_1$, we get that $\theta_1\, s_1 + (1-\theta_1) s_2 \in S$, for every $\theta_1 \in \mathbb{R}$; $e(\theta_1\, s_1 + (1-\theta_1)\, s_2) \in e(S) \subseteq S$. Linearity of e implies, $\theta_1\, e(s_1) + (1-\theta_1) e(s_2) \in S$ and we get that S is e-affine. $\square$

An e-affine set need not be a subspace of $\mathbb{R}^n$, we prove this by the following example.

Example 3.7. *Let* $S = \{(s, s+1) : s \in \mathbb{R}\}$ *and* $e : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be defined by* $e(s_1, s_2) = (-s_2, -s_1)$. *It can be verified that* e *is linear. Let* $(s_1, s_1+1), (s_2, s_2+1) \in S$ *and* $\theta \in \mathbb{R}$.

$$\begin{aligned}\theta e(s_1, s_1+1) + (1-\theta) e(s_2, s_2+1) &= \theta(-s_1-1, -s_1) + (1-\theta)(-s_2-1, -s_2)\\ &= (-\theta s_1 - \theta - s_2 - 1 + \theta s_2 + \theta, -\theta s_1 - s_2 + \theta s_2).\end{aligned}$$

Hence S *is* e*-affine and so* $e(S) \subseteq S$. *Since* $(0,0) \notin S$, S *is not a subspace of* $\mathbb{R}^n$.

For a linear, injective and idempotent map the converse of Proposition 3.6 holds.

Proposition 3.8. *Let* S *be an* e*-affine set containing the origin defined on a linear, injective and idempotent map* $e : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. *Then* S *is a subspace of* $\mathbb{R}^n$.

PROOF. Since S is e-affine, $\theta e(s_1) + (1-\theta)e(s_2) \in S$, $\theta \in \mathbb{R}$ and $s_1, s_2 \in S$. Let $s_2 = 0$, this implies that $e(s_2) = 0$. Hence, $\theta e(s_1) \in S$ and $e(\theta s_1) \in S$. Since e is idempotent, $e(\theta s_1) = e(e(\theta s_1)) \in e(S)$ so that $e(\theta s_1) = e(s)$, for some $s \in S$. Injectivity of e implies that $\theta s_1 = s \in S$. Now, we have

$$e\left(\frac{s_1+s_2}{2}\right) = \frac{1}{2}\, e(s_1+s_2) = \frac{1}{2}\, e(s_1) + \left(1 - \frac{1}{2}\, e(s_2)\right) \in S$$
$$e\left(\frac{s_1+s_2}{2}\right) = e\left(e\left(\frac{s_1+s_2}{2}\right)\right) \in e(S)$$

Since e is injective, $\frac{s_1+s_2}{2} \in S$. This implies that $s_1 + s_2 \in S$. Hence S is a subspace of $\mathbb{R}^n$. □

Definition 3.9. *Two e-affine sets S and T are parallel if $S = T + r$, for some $r \in \mathbb{R}^n$.*

Theorem 3.10. *Let $e : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invariant at $r \in \mathbb{R}^n$ and S be e-affine. Then $S + r$ is e-affine.*

PROOF. Let $s_1+r, s_2+r \in S+r$, where $s_1, s_2 \in S$ and $\theta \in \mathbb{R}$. Since S is e-affine and invariant at r,

$$\begin{aligned} \theta\, e(s_1+r) + (1-\theta)\, e(s_2+r) &= \theta\, (e(s_1)+r) + (1-\theta)\, (e(s_2)+r) \\ &= \theta . e(s_1) + (1-\theta) . e(s_2) + \theta r + (1-\theta) r \\ &= (\theta\, e(s_1) + (1-\theta)\, e(s_2)) + r \in S + r. \end{aligned}$$

Hence $S + r$ is e-affine. □

The above result establishes that translation of an e-affine is e-affine under certain constrain e.

Theorem 3.11. *In a linear map, the sum of two e-affine sets is e-affine.*

PROOF. Let $s_1 + s_2, s_3 + s_4 \in S_1 + S_2$ where $S_1, S_2 \subseteq \mathbb{R}^n$ be e-affine sets and $\theta \in \mathbb{R}$. Therefore $\theta\, e(s_1) + (1-\theta)\, e(s_3) \in S_1$ and $\theta\, e(s_2) + (1-\theta)\, e(s_4) \in S_2$. Since e is linear, we have $\theta\, e(s_1+s_2) + (1-\theta)\, e(s_3+s_4) \in S_1 + S_2$. Hence $S_1 + S_2$ is e-affine. □

Remark 3.12.
(i) *It can be easily verify that the linear combination of e-affine sets is e-affine whenever e is linear.*
(ii) *Also, for all $\alpha \in \mathbb{R}$, αS is e-affine whenever S is e-affine and e is linear.*

In (ii) the linearity of e cannot be lose one's grip on.

Example 3.13. *Consider $S = \{(s, s+1) : s \in \mathbb{R}\}$ and $e : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by $e(s_1, s_2) = (s_1, s_2 - 1)$ where $s_1, s_2 \in \mathbb{R}$. Choose $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \neq 1$. Then*

$$\begin{aligned} e(\alpha(s_1, s_2) + \beta(s_3, s_4)) &= e(\alpha s_1 + \beta s_3, \alpha s_2 + \beta s_4) \\ &= (\alpha s_1 + \beta s_3, \alpha s_2 + \beta s_4 - 1) \end{aligned}$$

and

$$\begin{aligned}\alpha e((s_1,s_2))+\beta e((s_3,s_4)) &= \alpha(s_1,s_2-1)+\beta(s_3,s_4-1)\\ &= (\alpha s_1+\beta s_3,\alpha s_2+\beta s_4-\alpha-\beta)\end{aligned}$$

and hence e is not linear. Fix $\alpha=-2$ and $(2,3),(-1,0)\in S$ which implies $(-4,-6),(2,0)\in\alpha S$. Then

$$\begin{aligned}s &= \theta e((-4,-6))+(1-\theta)e((2,0))\\ &= \theta(-4,-7)+(1-\theta)(2,-1)\\ &= (-4\theta,-7\theta)+(2-2\theta,-1+\theta)\\ &= (-6\theta+2,-6\theta-1)\end{aligned}$$

and put $\theta=-1$, $s=(8,5)=-2\left(-4,\frac{-5}{2}\right)\notin\alpha S$. Since $\left(-4,\frac{-5}{2}\right)\notin S$, which implies that αS is not e-affine.

Theorem 3.14. *For a linear map e, intersection of two e-affine sets is e-affine.*

Proof. The result follows from the definition. □

The above theorem holds for arbitrary intersection also. The upcoming example instantiate that union of two e-affine sets need not be e-affine.

Example 3.15. *Let $S_1=\{(s,s+1):s\in\mathbb{R}\}$, $S_2=\{(-s,s-1):s\in\mathbb{R}\}$ and $e:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be as $e(s_1,s_2)=(-s_2,-s_1)$. By Example 3.13, S_1 is e-affine. Let $(-s_1,s_1-1),(-s_2,s_2-1)\in S_2$ and $\theta\in\mathbb{R}$. Then*

$$\begin{aligned}\theta e((-s_1,s_1-1))+(1-\theta)e((-s_2,s_2-1)) &= \theta(-s_1+1,s_1)+(1-\theta)(-s_2+1,s_2)\\ &= (-\theta s_1+\theta-s_2+1+\theta s_2-\theta,\theta s_1+s_2-\theta s_2)\\ &= (-\theta s_1-s_2+\theta s_2+1,\theta s_1+s_2-\theta s_2)\\ &= (-(\theta s_1+s_2-\theta s_2-1),\theta s_1+s_2-\theta s_2)\in S_2.\end{aligned}$$

This proves that S_2 is e-affine. Also (0,1), (0,-1) $\in S_1\cup S_2$, we have

$$\begin{aligned}\theta e(0,1)+(1-\theta)e(0,-1) &= \theta(-1,0)+(1-\theta)(1,0)\\ &= (-\theta+1-\theta,0)\\ &= (1-2\theta,0)=(0,0)\quad \text{if}\quad \theta=\frac{1}{2}\notin S_1\cup S_2.\end{aligned}$$

This shows that $S_1\cup S_2$ is not e-affine.

Definition 3.16. *For an idempotent affine map $e:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, we define the following set*

$$S(e,a,\rho,\alpha)=\{s\in\mathbb{R}^n:<e(s),a>\rho\alpha\},$$

where $\alpha\in\mathbb{R},a\in\mathbb{R}^n$ and $\rho\in\{<\cdot,\cdot>,=,\leq,$ and $\geq\}$.

Theorem 3.17. *$S(e,a,\rho,\alpha)$ is e-affine.*

Proof. Let $s_1, s_2 \in S(e, a, \rho, \alpha)$ and $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
< e(\theta e(s_1) + (1-\theta)e(s_2)), a > &=< \theta e^2(s_1) + (1-\theta)e^2(s_2), a > \\
&=< \theta e(s_1) + (1-\theta)e(s_2), a > \\
&=< \theta e(s_1), a > + < (1-\theta)e(s_2), a > \\
&= \theta < e(s_1), a > +(1-\theta) < e(s_2), a > .
\end{aligned}$$

This implies that $e(s_1) + (1-\theta)\, e(s_2) \in S(e, a, \rho, \alpha)$. Thus $S(e, a, \rho, \alpha)$ is e-affine. □

Corollary 3.18. (i) *Any e-hyperplane is e-affine.*
(ii) *Any closed e-half-space is e-affine.*
(iii) *Any open e-half-space is e-affine.*

Corollary 3.19. *For a linear idempotent affine map e, the set $\{s \in \mathbb{R}^n :< e(s), a_i > \rho\alpha_i \, for \, i \in I\}$ is e-affine where $\rho\{< \cdot, \cdot >, =, \leq, \geq\}$.*

Theorem 3.20. *Let $S \subseteq \mathbb{R}^n$ be a non-empty e-affine set defined on a linear injective idempotent map e. Then S is parallel to a unique subspace of T.*

Proof. Since $S \neq \emptyset$ and e-affine, there is an element $e(s) \in S$. Now $S - e(s) = S + r$ is a translation of S where $r = -e(s)$. Also, $e(s) \in S, e(s) - e(s) = 0 \in S - e(s) = S + r$. Hence $S + r$ is a translation of S and $0 \in S + r$. Since e is linear and idempotent, $e(-e(s)) = e(y)$, so $e(r) = r$. Also, e is linear and has a fixed point at $r, S+r$ is e-affine, i.e., $S+r$ is e-affine containing the origin. Also, by Proposition 3.8, $S+r$ is a subspace of $\mathbb{R}^n$. Clearly S is parallel to a subspace T where $T = S+r$.

For the uniqueness, T_1 and T_2 are subspaces parallel to S. Then T_1 and T_2 are parallel to each other, i.e., $T_2 = T_1 + r_1$, for some $r_1 \in \mathbb{R}^n$. Again, T_2 is a subspace, $0 \in T_2$. Therefore $-r_1 \in T_1$ (i.e) $r_1 \in T_1$. Hence $T_2 = T_1 + r_1 \subseteq T_1$. By a similar argument we have $T_1 \subseteq T_2$. Thus $T_1 = T_2$. □

Theorem 3.21. *Let $e : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an idempotent affine map. Let S be an e-affine set in $\mathbb{R}^n$. Then $(\theta_1 + \theta_2)e(S) = \theta_1 e(S) + \theta_2 e(S)$, where $\theta_1, \theta_2 \in \mathbb{R}$ and $\theta_1 + \theta_2 \neq 0$.*

Proof. Let $s \in (\theta_1 + \theta_2)e(S)$. Obviously, $s \in \theta_1 e(S) + \theta_2 e(S)$. On the other hand, S is e-affine. This implies that $\frac{\theta_1}{\theta_1+\theta_2}e(S) + \frac{\theta_2}{\theta_1+\theta_2}e(S) \subseteq e(S)$, i.e., $\theta_1 e(S) + \theta_2 e(S) \subseteq (\theta_1 + \theta_2)e(S)$. □

4. e-affine cone

Definition 4.1. *A cone is a set S such that for any $s \in S$ and $\theta \geq 0, \theta s \in S$. Analogously we defined e-cone in $\mathbb{R}^n$. Any point s is of the form $s = \theta_1 s_1 + \theta_2 s_2, \theta_1, \theta_2 \geq 0$ is a conic combination of s_1 and s_2. A convex cone is a cone that is convex. That is a convex cone is a set contains all conic combinations $\{\theta_1 s_1 +$*

$\theta_2 s_2 + ... + \theta_n s_n | \theta_i \geq 0, i = 1, 2, ...n\}$ *of points in the set. In other words, it is closed under convex combinations.*

(i) *Any line is affine. Also when it passes through the origin, it is a convex cone.*
(ii) *Consider a ray $S = \{s + \theta s | \theta \geq 0\}$. If $s = 0$, then it is a convex cone.*

In this section, we initiate the notions of affine cone and e-affine cone.

Definition 4.2. *An affine (e-affine) cone is a cone (e-cone) that is affine (e-affine).*

Proposition 4.3. *Let e be a map such that $e(S) \subseteq S$, is closed under positive scalar multiplication and addition. Then S is an e-affine cone.*

Proof. Let $s_1, s_2 \in S$ and $\theta \in \mathbb{R}$. By hypothesis, $\theta e(s_1) + (1-\theta)e(s_2) \in e(S) \subseteq S$. That is S is e-affine. Also $\theta \in \mathbb{R}$, $\theta.e(s) \in e(S) \subseteq S$. This shows that S is e-cone and then S is an e-affine cone. □

Proposition 4.4. *For a linear idempotent map $e, e(S)$ is closed under positive scalar multiplication and addition where S is an e-affine cone.*

Proof. Let $s_1, s_2 \in S$ and $\theta \in \mathbb{R}$. Since S is e-affine, $\theta e(s_1) + (1-\theta)e(s_2) \in S$. Set $\theta = \frac{1}{2}$, $\frac{e(s_1)+e(s_2)}{2} \in S$. Since S is an e-cone, $2e\left(\frac{e(s_1)+e(s_2)}{2}\right) \in S$. This implies $e(s_1) + e(s_2) \in S$, because e is linear and idempotent. Consider $e(s_1) + e(s_2) = e^2(s_1) + e^2(s_2) = e(e(s_1) + e(s_2)) \in e(S)$, i.e., $e(S)$ is closed under addition. Then let $s \in S$ and $\theta \in \mathbb{R}$. As S is e-cone, so, $\theta e(s) \in S$ and linearity of e implies that $e(\theta s) \in S$. Then $e(\theta s) = e^2(\theta s) = e(e(\theta s)) \in e(S)$. □

Proposition 4.5. *If $\{S_i\}$ is an e-affine cone, then $\cap_{i\in I} S_i$ is also an e-affine cone.*

Proof. Let $S = \cap_{i\in I} S_i$ and $s \in S, \theta > 0$. Then $s \in S_i$, for all i. Each S_i is an e-affine cone, so, S_i is an e-cone. Then $\theta e(s) \in S_i$ for all i. This implies that $\theta e(s) \in \cap S_i = S$. Hence, S is an e-cone. Since each S_i is e-affine, S is also e-affine. □

5. Conclusion

An milestone of generalized e-convex sets that is e-affine has been introduced and explore some of its rudimentary properties. Also e-affine sets are characterized by linear and idempotent operators.

Acknowledgment

Dedicated this research article to all mathematicians working in this research field.

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