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Locally finite inverse semigroups

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Abstract. In this article, we study locally finite inverse semigroup S and characterize the structure of idempotents of S which are either a well-ordered countable chain or union of disjoint well-ordered countable chains. We also prove that when S is a locally finite Clifford semigroup, S is amenable if and only if minimal ideal of S is amenable.

1. Introduction

Let S denotes an inverse semigroup and $l^\infty(S)$ the usual Banach space of bounded complex function on S. Given $f \in l^\infty(S)$, $t \in S$, we define the translation functions $f \cdot t,\ t \cdot f$ by

$$f \cdot t(s) = f(ts),\ t \cdot f(s) = f(ts)\ (s \in S).$$

A continuous linear functional μ on $l^\infty(S)$, i.e. an element $\mu \in l^\infty(S)^*$ is said to be left (right) invariant if, for all $f \in l^\infty(S),\ t \in S$,

$$\mu(f \cdot t) = \mu(f),\ (\mu(t \cdot f) = \mu(f)).$$

Also, μ is a mean if $\mu(\mathbf{1}) = \|\mu\| = 1$, where $\mathbf{1}$ denotes the constant unite function on S. The semigroup S is left (right) amenable if there exists a left (right) invariant mean on $l^\infty(S)$. Since the mapping $s \to s^*$ is an involution on S, i.e. $ss^*s = s$, $(st)^* = t^*s^*$ for all $s, t \in S$, it follows that S is left amenable if and only if it is right amenable. For more information on amenable and character inner amenable semigroups, we refer to [2] and [1], respectively.

We show that amenability of S and S/ξ is equivalent, where ξ is maximal Clifford homomorphic image of S. Duncan and Namioka have shown in [3] that the inverse semigroup S is amenable if and only if the maximal group homomorphic image G_S of S is amenable. We also prove for certain class of Clifford semigroup S, E is a well-ordered chain and in particular it is countable. This fails to be true for each Clifford semigroup. In addition, we show that the idempotents of locally finite inverse semigroup S satisfies minimal condition property and so have a minimal

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idempotent. Idempotents subsemigroup of locally finite semigroup of S is union of well-ordered chains and each Clifford semigroup with this set as idempotents sub semigroup is amenable if and only if maximal subgroup contains minimal idempotent is amenable.

2. Amenability of semigroups

We firstly give some definitions and basic properties of semigroups that we shall need. The standard reference for the theory of semigroups is [5].

Definition 2.1. The semigroup S is an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^2$.

Throughout this paper, S is an inverse semigroup with the set of idempotents E. An inverse semigroup whose idempotents are in the center is called a Clifford semigroup.

An equivalence relation R on a semigroup S is called a congruence if

$$(s,t) \in R \Rightarrow (as,at),\ (sa,ta) \in R\ (s,\ t,\ a \in S).$$

Congruences on any semigroup provide some information about its homomorphic images [4].

Let ρ be a congruence on S and P a property of homomorphic image S/ρ, we call ρ a P congruence. A least congruence ρ such that S/ρ is a P congruence is called the least P congruence.

Let σ be the congruence on S defined by $s\sigma t$ if there exists $e \in E$ with $es = et$. The quotient semigroup S/σ is then a group and and σ is the least group congruence on S. The least Clifford congruence here is denoted by ξ.

Theorem 2.1. *The inverse semigroup S is amenable if and only if S/ξ is amenable.*

Proof. Let $\chi : S \to S/\xi$ be the canonical homomorphism and S be amenable. Any homomorphic image of S is amenable, in particular S/ξ is amenable. Conversely, let S/ξ be amenable. Since each group congruence is a Clifford congruence, we have $\xi \subseteq \sigma$ and so $S/\xi/\sigma \cong S/\sigma$. Thus S/σ is a homomorphic image of S/ξ and so it is amenable. Now amenability of S follows from amenability of S/σ, by Theorem 1 of [3]. □

By [5], for each inverse semigroup of S, there is a partial order on S defined by

$$s \le t \Longleftrightarrow s = ss^*t\ (s,t \in S).$$

A partially ordered set $(S, \le)$ is called locally finite if $(s] = \{t \in S :\ t \le s\}$ is finite for each $s \in S$. We say that $(S, \le)$ is uniformly locally finite if $sup\{|(s]| : s \in S\} < \infty$. Throughout, by (uniformly) locally finite inverse semigroup we mean a (uniformly) locally finite inverse semigroup with respect to the partial ordering. Recall that the set $Ee = \{i \in E : \ i \leq e\}$ is a principal ideal of E and semilattice E is uniform if

$$Ee \simeq Ef \ (e, f \in E),$$

and semilattice E is anti-uniform if, for all $e, f \in E$,

$$Ee \simeq Ef \Rightarrow \ e = f.$$

We shall say that $(X, \leq)$ satisfies the minimal condition if every nonempty subset of X has a minimal element. A totally ordered set X satisfying the minimal condition is said to be well-ordered.

Theorem 2.2. *Let S be a semilattice. Then S is locally finite if and only if S is a countable (finite) well ordered chain of idempotents or it contains well ordered chains of idempotents which for distinct chain of E_i, E_j we have $E_i \cap E_j = e, E_iE_j = \{e\}$, where e is minimal idempotent of S.*

Proof. We claim that S satisfies minimal condition. Let A be a nonempty subset of S. We have $\{ef : \ f \in A\} \subseteq (e]$, for each $e \in A$. Since S is locally finite, S is locally finite and so $\{ef : \ f \in A\}$ is finite. Also $\{ef : \ f \in A\} \cap A \neq \emptyset$ and is finite. Thus A has a minimal element. Thus S satisfies minimal condition and so S has a minimal idempotent e. Now suppose that S is totally ordered. It is clear that S contains a countable chain

$$e_1 = e \leq e_2 \leq e_3 \leq \cdots ,$$

which for each $i > 1$, e_i is minimal element of the set $\{g \in E : \ g \geq e_{i-1}, \ g \neq e_{i-1}\}$. Now suppose that $\{e \in E : \ e \neq e_i \ \textit{for some} \ i\} \neq \emptyset$. Therefore it has a minimal element f. It follows from locally finiteness of S that $\{e_i : \ e_i \leq f\}$ is finite. Hence it has a maximum element e_j. Thus $e_{j+1} \geq f$ and since $f \geq e_j$ and $f \neq e_j$, It contradicts selection of e_{j+1}. Thus S is the countable chain $e_1 \leq e_2 \leq \cdots$. Now suppose that S is not totally ordered set. Similarly S contains at least a countable chain E_1. Since S satisfies minimal condition, $S \backslash \{E_1 \backslash \{e\}\}$ has a minimal idempotent f_2. Clearly $f_2e_2 = e_1$, we claim that for each $j > 2$, $f_2e_j = e_1$. Otherwise,

$$\begin{aligned} f_2e_j = e_t \ (1 < t < j) \ &\Rightarrow f_2e_je_2 = e_te_2 \\ &\Rightarrow f_2e_2 = e_2, \end{aligned}$$

which is a contradiction. Similarly S contains countable chain E_2

$$e_1 = e \leq f_2 \leq f_3 \leq \cdots ,$$

which $f_ie_j = e_1$. By similar way we have the result. □

Example 2.2. Let ϵ be a one-one map from $\mathbb{Q}$ onto the set $\mathbb{N}^0 = \{0, 1, 2, \dots\}$ of non-negative integers, and let

$$E = \bigcup_{q \in \mathbb{Q}} \{(q, 0), (q, 1), \dots, (q, q\epsilon)\},$$

such that

$$(q, n) \cdot (p, m) = \begin{cases} (p, m) & p < q \\ (q, n) & q < p \\ (p, \min(m, n)) & p = q. \end{cases}$$

Howie and Schein showed that E has the property that every inverse semigroup with E as semilattice of idempotents is Clifford but it is not a well ordered chain [6].

Example 2.3. Let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation, that is $m \vee n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent and $\mathbb{N}$ is a well-ordered chain but it fails to be locally finite.

Example 2.4. Let $I = \mathbb{N}$, $G = \{e\}$ and $E = \mathcal{M}^0(G, I)$ be the Brandt semigroup over G with index set I.
The multiplication operation is defined by

$$(e)_{ij}(e)_{kl} = \begin{cases} (e)_{il} & if\ j = k \\ 0 & if\ j \neq k. \end{cases}$$

Clearly E is not a well ordered chain but it is locally finite and it satisfies minimal condition.

Theorem 2.3. *Let $\varphi : S \to T$ be a homomorphism of (uniformly) locally finite semigroup S onto semigroup T. Then T is (uniformly) locally finite.*

Proof. Let $\varphi(f)$ be an element of $(\varphi(e)]$ such that $\varphi(f) \neq \varphi(e)$. Then $\varphi(fe) = \varphi(f)$. Since $fe \in (e]$ and $|(e]| \leq \sup\{|(x]| : x \in S\} < \infty$, we have $|(\varphi(e)]| \leq |\{fe : f \in E\}| \leq \sup\{|(x]| : x \in S\} < \infty$, and so T is also (uniformly) locally finite semigroup. □

Corollary 2.4. *Let S be a (uniformly) locally finite inverse semigroup. If S is (uniformly) locally finite then S/ξ is (uniformly) locally finite.*

Proof. Consider the canonical homomorphism $\pi : S \to S/\xi$ and the above theorem. □

Proposition 2.5. *Let S be a locally finite inverse semigroup with minimal idempotent e. Then S is amenable if and only if the maximal sub groups of S containing e is amenable.*

Proof. By Theorem 2.2, the maximal sub groups of S containing e is minimal ideal of S and so S is amenable if and only if G_e is amenable. □

Theorem 2.6. *Let $\varphi : S \to T$ be a homomorphism of the Clifford semigroup S onto the locally finite inverse semigroup T such that $f\varphi^{-1}$ is a finite subsemigroup of S for each idempotent element $f \in T$. Then S is locally finite.*

PROOF. For each $e \in E_S$ and $g \in (e]$, we have $g \in \cup_{i=1}^{n} \varphi^{-1}(f_i)$, which $f_i \in (\varphi(e)]$. If $(e]$ is infinite, then there is $1 \leq i \leq n$ such that $|(e] \cap \varphi^{-1}(f_i)| = \infty$. But since $\varphi^{-1}(f_i)$ is finite, we have $(e] \cap \varphi^{-1}(f_i)$ is finite. Thus $(e]$ is finite and S is locally finite. □

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