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The generalized Hyers-Ulam stability of derivations in non-Archimedean Banach algebras

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Abstract. In this paper, the generalized Hyers-Ulam stability of the functional inequality

$$\|f(a)+f(b)+cf(d)+f(c)d\| \leq \left\|kf\Big(\frac{a+b+cd}{k}\Big)\right\|, \qquad |k|<|2|,$$

in non-Archimedean Banach algebras is established.

1. Introduction

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot|:\mathbb{K}\longrightarrow[0,\infty)$ such that for any $a,b\in\mathbb{K}$

(i) $|a|\geq 0$ and equality holds if and only if $a=0$,

(ii) $|ab|=|a||b|$,

(iii) $|a+b|\leq max\{|a|,|b|\}$.

Condition (iii) is called the strict triangle inequality. By (ii), we have $|1|=|-1|=1$. Thus, by induction, is concluded from (iii) that $|n|\leq 1$ for each integer n. In all, we always assume that $|\cdot|$ is non-trivial, i.e., that there is an $a_0\in\mathbb{K}$ such that $|a_0|\notin\{0,1\}$. Throughout this paper, we assume that the base field is a non-Archimedean valuation field.

Definition 1.1. Let $\mathcal{X}$ be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|.\|:\mathcal{X}\longrightarrow\mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(I)$\|x\|=0$ if and only if $x=0$;

(II) $\|rx\|=|r|\|x\|$ for all $r\in\mathbb{K}$ and $x\in\mathcal{X}$;

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(III) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq max\{\|x\|, \|y\| \quad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|.\|)$ is called a non-Archimedean normed space.

It is concluded from (III) that

$$\|x_m - x\| \leq max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\} \quad (m > l).$$

Therefore, a sequence $\{x_m\}$ is Cauchy in $\mathcal{X}$ if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. S. M. Ulam posed a number of important unsolved problems during his talk at the university of Wisconsin in 1940 [**15, 16**]. One of his open problems was the first stability problem as follows:

Let $\mathcal{G}_1$ be a group and let $(\mathcal{G}_2, d)$ be a metric group. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathcal{G}_1$, then there exists a homomorphism $T : \mathcal{G}_1 \to \mathcal{G}_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in \mathcal{G}_1$?

Hyers [**5**] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [**1**] for additive mappings and by Rassias [**13**] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [**13**] has had a lot of influence in the development of what we call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations [**2, 14**]. Since then many mathematicians have been working on this area of research [**3, 4**]. The stability of derivations was studied by C.-G. Park [**12**] and Moslehian [**6**]. Beginning around the year 1980, the topic of approximate homomorphisms and derivations and their stability theory in the field of functional equation and inequalities was taken up by several mathematicians. Also, the stability problems in non-Archimedean Banach spaces(algebras) are studied by Moslehian and Rssias [**7**], Moslehian and Sadeghi [**8, 9**], Mirmostafaee [**10**] and Najati and Moradlou [**11**]. In this paper, we prove that if f satisfies the functional inequality

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\| kf\Big(\frac{a+b+cd}{k}\Big) \right\|, \qquad |k| < |2|, \tag{1}$$

then f is a derivation. Moreover, we prove the generalized Hyers-Ulam stability of functional inequality (1) in non-Archimedean Banach algebras.

2. Main Results

Throughout this section, $\mathcal{A}$ is a non-Archimedean Banach algebra on non-Archimedean filed $\mathbb{K}$ that the characteristic of $\mathbb{K}$ is not 2 and $\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule.

Proposition 2.1. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $\mathcal{A}$ be a unital non-Archimedean Banach algebra, $\mathcal{X}$ be a non-Archimedean Banach $\mathcal{A}$-bimodule and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be a mapping such that*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\Big(\frac{a+b+cd}{k}\Big)\right\| \tag{2}$$

for all $a, b, c, d, \in \mathcal{A}$. Then f is a derivation.

Proof. Taking $a = b = c = d = 0$ in (2) we have $\|2f(0)\| \leq \|kf(0)\|$ and So by $|k| < |2|$ we get $(|2| - |k|)\|f(0)\| \leq 0$. Therefore, $f(0) = 0$. We now show that f is an odd function. Set $b = -a$ and $c = d = 0$ in (2). Hence, we have $\|f(a) + f(-a)\| \leq |k|\|f(0)\|$ and so $f(-a) = -f(a)$ for all $a \in \mathcal{A}$.

In this step we show that $f(a^2) = af(a) + f(a)a$ for all $a \in \mathcal{A}$ and therefore we can conclude $f(1) = 0$. For this purpose it is enough to take $a := a^2, b := 0, c := -a$ and $d := a$ in (2). Thus, we get

$$\|f(a^2) + f(0) - af(a) + f(-a)a\| \leq \left\|kf\Big(\frac{a^2+0-aa}{k}\Big)\right\| = 0.$$

Now, letting $c := 1$ and $d := -a - b$ in (2), we have

$$\|f(a) + f(b) - f(a+b)\| \leq \left\|kf\Big(\frac{a+b-(a+b)}{k}\Big)\right\| = 0.$$

As a result, we have $f(a+b) = f(a)+f(b)$. In the last step set $a := ab, b := 0, c := -a$ and $d := b$ in (2) and so we can see that

$$\|f(ab) + f(0) - af(b) - f(a)b\| \leq \left\|kf\Big(\frac{ab+0+a(-b)}{k}\Big)\right\| = 0.$$

Therefore, $f(ab) = af(b) + f(a)b$ and this completes the proof. □

Theorem 2.2. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $r < 1$, θ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\begin{aligned}\|f(a) + f(b) + cf(d) + f(c)d\| \leq &\left\|kf\Big(\frac{a+b+cd}{k}\Big)\right\| \\ &+ \theta(\|a\|^r + \|b\|^r + \|cd\|^r)\end{aligned} \tag{3}$$

for all $a, b, c, d \in \mathcal{A}$. Then, there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that

$$\|f(a) - D(a)\| \leq \frac{2+|2|^r}{|2|^r}\theta\|a\|^r \qquad (a \in \mathcal{A}). \tag{4}$$

Proof. Set $a = b, c = -2a$ and $d = 1$ in(3). We have

$$\begin{aligned}\|f(a) + f(a) + (-2a)f(1) + f(-2a)\| \leq &\left\|kf\Big(\frac{a+a+(-2a)}{k}\Big)\right\| \\ &+ \theta(\|a\|^r + \|a\|^r + \|2a\|^r)\|\end{aligned} \tag{5}$$

Therefore we can conclude that $\|2f(a) - f(2a)\| \leq (2 + |2|^r)\theta\|a\|^r$ and so $\|f(a) - 2f(\frac{a}{2})\| \leq \Big(\frac{2+|2|^r}{|2|^r}\Big)\theta\|a\|^r$. Now, by induction we get

$$\Big\|2^{n+1}f\Big(\frac{a}{2^{n+1}}\Big) - 2^n f\Big(\frac{a}{2^n}\Big)\Big\| \leq \Big(\frac{2+|2|^r}{|2|^{(r-1)n+1}}\Big)\theta\|a\|^r. \tag{6}$$

It follows from (6) that the sequence $\Big\{2^n f\Big(\frac{a}{2^n}\Big)\Big\}$ is a Cauchy sequence for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is a non-Archimedean Banach algebra, the sequence $\Big\{2^n f\Big(\frac{a}{2^n}\Big)\Big\}$ converges. Hence, one can define mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ by $D(a) := \lim_{n\to\infty} 2^n f\Big(\frac{a}{2^n}\Big)$ for all $a \in \mathcal{A}$. It follows from (3) and definition of D that

$$\begin{aligned}
&\|D(a) + D(b) + cD(d) + D(c)d\|\\
&= \lim_{n\to\infty} \|4^n f\Big(\frac{a}{4^n}\Big) + 4^n f\Big(\frac{b}{4^n}\Big) + c2^n f\Big(\frac{d}{2^n}\Big) + f\Big(\frac{c}{2^n}\Big)2^n d\|\\
&= \lim_{n\to\infty} \Big\|4^n\Big(f\Big(\frac{a}{4^n}\Big) + f\Big(\frac{b}{4^n}\Big) + \frac{c}{2^n}f\Big(\frac{d}{2^n}\Big) + f\Big(\frac{c}{2^n}\Big)\frac{d}{2^n}\Big)\Big\|\\
&\leq \lim_{n\to\infty} \Big\|4^n k f\Big(\frac{a+b+cd}{4^n k}\Big)\Big\|\\
&+ \lim_{n\to\infty} \frac{|4|^n}{|4|^{nr}}\theta(\|a\|^r + \|b\|^r + \|cd\|^r) = \Big\|kD\Big(\frac{a+b+cd}{k}\Big)\Big\|
\end{aligned} \tag{7}$$

and so $\|D(a) + D(b) + cD(d) + D(c)d\| \leq \Big\|kD\Big(\frac{a+b+cd}{k}\Big)\Big\|$. By Proposition 2.1, the mapping $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a derivation. Now it is enough to prove D is unique. Let $D' : \mathcal{A} \longrightarrow \mathcal{X}$ be another derivation satisfying (4). Then, we have

$$\begin{aligned}
\|D(a) - D'(a)\| &= \Big\|2^n D\Big(\frac{a}{2^n}\Big) - 2^n D'\Big(\frac{a}{2^n}\Big)\Big\|\\
&\leq max\Big\{\Big\|2^n D\Big(\frac{a}{2^n}\Big) - 2^n f\Big(\frac{a}{2^n}\Big)\Big\|, \Big\|2^n D'\Big(\frac{a}{2^n}\Big) - 2^n f\Big(\frac{a}{2^n}\Big)\Big\|\Big\}\\
&\leq \frac{2+|2|^n}{|2|^{(r-1)n+1}}\theta\|a\|^r
\end{aligned} \tag{8}$$

which tends to zero as $n \to \infty$ for all $a \in \mathcal{A}$. So $D(a) = D'(a)$ for all $a \in \mathcal{A}$. □

Theorem 2.3. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $r > 1$, θ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\begin{aligned}
\|f(a) + f(b) + cf(d) + f(c)d\| \leq \Big\|kf\Big(\frac{a+b+cd}{k}\Big)\Big\|\\
+ \theta(\|a\|^r + \|b\|^r + \|cd\|^r)
\end{aligned} \tag{9}$$

for all $a, b, c, d \in \mathcal{A}$. Then, there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that

$$\|f(a) - D(a)\| \leq \frac{2+|2|^r}{|2|}\theta\|a\|^r \qquad (a \in \mathcal{A}). \tag{10}$$

Proof. Using the proof method of previous theorem, we can see that

$$\|f(a) - \frac{1}{2}f(2a)\| \leq \left(\frac{2 + |2|^r}{|2|}\right)\theta\|a\|^r. \tag{11}$$

The rest of the proof of is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $r < \frac{1}{3}$, θ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta\|a\|^r\|b\|^r\|cd\|^r \tag{12}$$

for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that

$$\|f(a) - D(a)\| \leq \frac{\theta|2|^r}{|2|^{3r}}\|a\|^{3r} \qquad (a \in \mathcal{A}). \tag{13}$$

Proof. Letting $a = b := a, c := -2a$ and $d = 1$ in the (12), we obtain

$$\|f(a) + f(a) - 2af(1) + f(-2a)1\| \leq \left\|kf\left(\frac{a+a-2a}{k}\right)\right\| + \theta\|a|^r\|a\|^r\|2a\|^r$$
$$\|2f(a) - f(2a)\| \leq \theta\|a\|^{3r}|2|^r$$

and so

$$\left\|f(a) - 2f\left(\frac{a}{2}\right)\right\| \leq \frac{|2|^r\theta}{|2|^{3r}}\|a\|^{3r}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.5. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $r > \frac{1}{3}$, θ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta\|a\|^r\|b\|^r\|cd\|^r) \tag{14}$$

for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\|f(a) - D(a)\| \leq \frac{\theta|2|^r}{|2|}\|a\|^{3r} \qquad (a \in \mathcal{A}). \tag{15}$$

Proof. Letting $a = b := a, c := -2a$ and $d = 1$ in the (14), we can obtain

$$\|2f(a) - f(2a)\| \leq \theta\|a\|^{3r}|2|^r$$

and so

$$\Big\|f(a)-\frac{1}{2}f\Big(\frac{a}{2}\Big)\Big\|\leq\frac{|2|^r\theta}{|2|}\|a\|^{3r}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

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