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A simple method to solve nonlinear Volterra-Fredholm integro-differential equations

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Abstract. In this paper, a new simple direct method to solve nonlinear Fredholm-Volterra integral equations is presented. By using Block-pulse (BP) functions, their operational matrices and Taylor expansion a nonlinear Fredholm-Volterra integral equation converts to a nonlinear system. Some numerical examples illustrate accuracy and reliability of our solutions. Moreover, the effect of noise shows our method is stable.

1. Introduction

Integral equations are widely employed in many fields of mathematics and sciences such as physics, engineering, continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics and radiation, the particle transport of astrophysics and reactor theory, fluid mechanics and so forth. Therefore, finding an acceptable solution to such equations is necessary. Hence, as to solve integral equations, researchers have presented different methods among which one can make mention of analytical numerical and mixed methods. See [1, 2, 3, 4, 5, 6, 7]. In this paper, we consider a nonlinear Fredholm-Volterra integral equation as follows:

$$\begin{cases} l\left[u(x)\right] = f(x) + \lambda_1 \int_a^b k_1(x,t)G(u(t))dt + \lambda_2 \int_a^x k_2(x,t)H(u(t))dt \\ u^{j}(a) = v_i, \qquad j = 0, \dots, n-1. \end{cases} \tag{1}$$

where $G(t), H(t)$ are smooth functions and u is unknown function and n is a positive integer number and $\lambda_1, \lambda_2 \in \mathbb{R}$. Moreover, $\ell^2\left[a,b\right])$ or $\ell^2([a,b] \times [a,b]$. In addition, l is a linear differential operator as follows:

$$l\left[u(x)\right] = a_n u^n + a_{n-1}u^{n-1} + \ldots + a_1 u' + a_0 u, \tag{2}$$

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where $a_n \in \mathbb{R}$ for $i = 0, 1, \ldots, n$. The following theorem allows us to these functions in our approximation.

Theorem 1.1. [1] *Suppose that* $H = \ell^2[a, b]$ *is a Hilbert space with the inner product that is defined by* $\langle f, g\rangle = \int_a^b f(t)g(t)dt$ *and* $Y = Span\{y_1, y_2, \ldots, y_m\}$. *Let* f *be an arbitrary element in* H. *Since* Y *is a finite dimensional and closed subspace, it is a complete subset of* H *and so* f *has the unique best approximation out of* Y.

Many researchers studied and discuss the linear Volterra-Fredholm integro- differential equations. E. Boabolian, Z. Masouri and S. Hatamazadeh Varmazyar [2] in 2008 construct new direct method to solve non-linear Volterra-Fredholm integral and integro-differential equation using operational matrix block-pulse functions. A. ALJubory [3] in 2010 introduced some approximation method for solving Volterra-Fredholm integral and integro-differential equation. M. Dadkah, M. Tavassoli Kajanj and S. Mahdavi [4] in 2010 used numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Legendre wavelets. M. Rabani and S. H. Kiasoltani [5] in 2011 study the solving of non-linear system of Volterra-Fredholm integro-differential equation by using discrete collocation method. H. D. Gherjalar and M. Hossein [6] in 2012 solved integral and integro-differential equation by using B-splines function.

Using a sequence of polynomials or functions is common to solve integro-differential equations and integral equations. In these methods, integro-differential equations convert to a system. Solve the system gives us to solution of integro-differential equation.

In this paper, we use Block-pulse functions and their properties. Then operational matrices help us to convert integro-differential equation to a nonlinear system. For nonlinear terms in integro-differential equation we use Taylor expansion. Block-pulse functions properties caused to have a nonlinear system that can be solved easily.

2. Block-pulse functions

BP functions are famous functions that many authors used them to solve different equations.

Definition 2.1. Suppose m is a positive integer number, an m-set of BPFs defined over $[0, T)$ as [7]:

$$\phi_i(t) = \begin{cases} 1, & \frac{iT}{m} \leqslant t < \frac{(i+1)T}{m}, \\ 0, & otherwise. \end{cases}$$

where $i = 0, 1, \ldots, m-1$ and $\phi_i(t)$ is the ith BPF and consider $h = \dfrac{T}{m}$.

In this paper, for convince we consider our interval $[0,1)$. We denote $\Phi_m(t)$ as an m-vector as follows.

$$\Phi_m(t) = \begin{bmatrix}\phi_0(t) & \phi_1(t) & \dots & \phi_{m-1}(t)\end{bmatrix}^T, t \in [0,1). \tag{3}$$

It is easy to see, the BPFs have many properties that most important of them are disjointness, orthogonality, and completeness [7].

$$(P1)\phi_i(t).\phi_j(t) = \begin{cases} 0, & i \neq j, \\ \phi_i(t), & i = j. \end{cases}$$

$$(P2)\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \text{where } \delta_{ij} \text{ is the Kroneker delta.}$$

(P3) For every $f \in \ell^2[0,1]$ when $m \to \infty$, Parsevala's identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \left\|\phi_i(t)\right\|^2,$$

where $f_i = \frac{1}{h}\int_0^1 f(t)\phi_i(t)dt$.

(P4) Let V be an m-vector. Then

$$\Phi_m(t)\Phi_m^T(t)V = \tilde{V}\Phi_m(t) \tag{4}$$

where $\tilde{V}$ is an $m \times m$ diagonal matrix such that $\tilde{V}_{ii} = V_i$, for $i = 0, 1, \dots, m-1$.

(P5) For every $m \times m$ matrix B:

$$\Phi_m^T(t)B\Phi_m(t) = \hat{B}\Phi_m(t), \tag{5}$$

where $\hat{B}$ is an m-vector such that $\hat{B}_i = B_{ii}$, for $i = 0, 1, \dots, m-1$.

3. BPFs expansion and operational Matrices

Consider $f \in \ell^2[0,1]$, with respect to BPFs on $[0,1)$ and we can write

$$f(t) = \sum_{i=0}^{\infty} f_i\phi_i(t) \simeq \sum_{i=0}^{m} f_i\phi_i(t) = F^T\Phi_m(t) = \Phi_m^T F, \tag{6}$$

where $F = \begin{bmatrix} f_0 & f_1 & \dots & f_{m-1}\end{bmatrix}^T$ and f_i is defined as (P3).

Theorem 1.1 guarantees uniqueness of coefficients. Now, assume $k(x,t) \in (\ell^2[0,1) \times \ell^2[0,1))$ is a two dimensional function. With respect to BPFs we can write

$$k(x,t) \simeq \Phi_m^T K \Phi_m, \tag{7}$$

where $K_{ij} = m^2 \int_0^1 \int_0^1 k(x,t)\phi_i(x)\phi_j(t)dxdt, \quad i,j = 0,1,\ldots,m-1$.

Here, we obtain operational matrix of dual.

Lemma 3.1. *Let m be an integer and $\Phi_m(t)$ defined as (3). Then*

$$\int_0^1 \int_0^1 \Phi_m(x)\Phi_m^T(t)dx = hI_m. \tag{8}$$

Proof. With respect to $(P2)$, the result is obvious. □

Now we compute operational matrix of integration.

Lemma 3.2. *Let $0 \leq x \leq 1$ and $\Phi_m(t)$ defined as (3). Then*

$$\int_0^x \Phi_m(t)dt \simeq P\Phi_m(x), \tag{9}$$

where P, the operational matrix of integration, is an $m \times m$upper triangular matrix and can be presented as $P = \frac{h}{2}\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$.

Proof. See [7]. □

4. Main idea

We firstly note that if $f \in \ell^2[0,1)$, then $\int_0^x f(t)dt = F^T P\Phi_m(x)$. Furthermore, for derivate of $f(x)$ we have

$$F' = (P^{-1})^T (F - F_0),$$

whereF' is operational matrix of derivative, F_0 is expansion of $f(0)$.
With apply mathematical induction we can write as

$$F^{(n)} = D^n F - D^n F_0 - \cdots - DF_0^{(n-1)} = D^n F - \sum_{i=0}^{n-1} D^{(n-i)} F_0^{(i)} \tag{10}$$

where $D = (P^{-1})^T$ and $F_0^{(i)}$ is is expansion of $f^i(0)$ with respect to BPFs. For linear differential operator operational matrix, it is enough to use above equality for differential operator terms. Apply (10) for the linear differential operator are shown in (2), we can write

$$D^n U - \sum_{i=0}^{n-1} D^{(n-i)} U_0^{(i)} + a_{n-1}(D^{n-1}U - \sum_{i=0}^{n-2} D^{(n-i-1)} U_0^{(i)}) + \cdots + a_1 D(U - U_0) + a_0 U$$
$$= lDU + R,$$

where R is remaining terms.

Lemma 4.1. *Let $f(x), g(x) \in \ell^2[0,1)$. Then, relation (3) implies $f(x)g(x) = H^T\Phi_m(x)$, where*

$$H = \begin{bmatrix} f_0g_0 & f_1g_1 & \cdots & f_{m-1}g_{m-1} \end{bmatrix}^T, F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{m-1} \end{bmatrix}^T$$

and

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{m-1}. \end{bmatrix}^T$$

As a result, if $f(x) \in \ell^2[0,1)$, then $f^n(x) = H^T\Phi_m(x)$, where

$$H = \begin{bmatrix} f_0^n & f_1^n & \cdots & f_{m-1}^n \end{bmatrix}^T.$$

Now, if $\psi(x) \in C^\infty[0,1]$, then $\psi(x)$ has Taylor expansion as follows:

$$\psi(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Suppose $u(x) \in \ell^2[0,1)$ is an arbitrary function with respect to (3). We can write $u(x) = U^T\Phi_m(x) = \Phi_m^T(x)U$, where $U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \end{bmatrix}^T$. Now

$$\psi(u(x)) = \sum_{i=0}^{\infty} a_i u^i(x) \simeq \sum_{i=0}^{N} a_i u^i(x).$$

With respect to (3), we have $\psi(u(x)) = \Psi^T\Phi_m(x)$, therefore according to what was said

$$\Psi_j = \sum_{i=0}^{N} a_i u_j^i(x), \quad j = 0, 1, ..., m-1.$$

5. Direct method to solve nonlinear Volterra-Fredholm integro-differential equation

The results mentioned in previous sections are used to obtain a direct efficient method to solve nonlinear Volterra-Fredholm integro-differential equation. Consider the nonlinear Volterra-Fredholm integro-differential equation which are shown in (1), where $u(x) \in \ell^2[0,1)$ is unknown, $k_1(x,t), k_2(x,t) \in \ell^2([0,1) \times [0,1))$ are known and $G(x), H(x) \in C^\infty[0,1)$. Moreover, λ_1, λ_2 are two real parameters. Here, we approximate Fredholm term, with respect to (5) and (6) as

$$k_1(x,t) = \Phi_m^T(x)K_1\Phi_m(t), G(u(t)) = \Phi_m^T(t)G_u.$$

Hence, (7) indicates that

$$\int_0^1 \Phi_m^T(x)K_1\Phi_m(t)\Phi_m^T(t)G_u dt = h\Phi_m^T(x)K_1G_u,$$

where $[G_u]_j = \sum_{i=0}^{N} a_i u_j^i(x), \quad j = 0, 1, ..., m-1$. For the Volterra term we use (5) and (6), we can write

$$k_2(x,t) = \Phi_m^T(x)K_2\Phi_m(t), H(u(t)) = \Phi_m^T(t)H_u.$$

Now with respect to (3), (6) and (8) we can see

$$\int_0^x \Phi_m^T(x)K_2\Phi_m(t)\Phi_m^T(t)Hdt = \Phi_m^T(x)K_2\tilde{H}\int_0^x \Phi_m(t)dt = \Phi_m^T(x)K_2\tilde{H}_uP\Phi_m.$$

Put $H_u = K_2\tilde{H}_uP$, and so Volterra term has the following matrix form:$\Phi_m^T(x)\hat{H}_u$. The final matrix form of nonlinear Volterra-Fredholm integro-differential equation is as follows.

$$\ell[D]U - \lambda_1 hK_1G_u - \lambda_2\hat{H}_u = F + R.$$

We are going to apply our method to some numerical examples. We selected examples from different references, so our results can be compared with the results from other methods. Results of examples are shown in related tables. We note that in tables $e_{m,N}$ indicates absolute error of our approximation with respect to m and N.

Example 5.1. Consider following nonlinear Fredholm-Volterra integral equation of the second kind with the exact solution $u(x) = 1 - x$.

$$\begin{aligned} u(x) &= \frac{1}{12}(19 - 28x - 6\sin 1\sin x - 6x\cos 1\sin x + 6\sin 1\cos x) \qquad (11) \\ &+ \int_0^x \sin(x-t)\cos(u(t))dt + \int_0^1 (1+u^2(t))(x-t)dt \end{aligned}$$

Results are shown in table 1.

x	$m=50, N=10$	$e_{50,10}$	$m=100, N=10$	$e_{100,10}$	exact solution
0.00	0.990	1.00×10^{-2}	0.995	5.0×10^{-3}	1.0
0.1	0.890	1.00×10^{-2}	0.895	5.0×10^{-3}	0.9
0.2	0.7899	1.01×10^{-2}	0.795	5.0×10^{-3}	0.8
0.3	0.6899	1.01×10^{-2}	0.695	5.0×10^{-3}	0.7
0.4	0.5899	1.01×10^{-2}	0.595	5.0×10^{-3}	0.6
0.5	0.4899	1.01×10^{-2}	0.495	5.0×10^{-3}	0.5
0.6	0.3899	1.01×10^{-2}	0.395	5.0×10^{-3}	0.4
0.7	0.2899	1.01×10^{-2}	0.295	5.0×10^{-3}	0.3
0.8	0.1899	1.01×10^{-2}	0.195	5.0×10^{-3}	0.2
0.9	0.0899	1.01×10^{-2}	0.095	5.0×10^{-3}	0.1

Table 1: Results of example 1

Example 5.2. Suppose $u(x) = x$ be the exact solution of the following nonlinear Fredholm-Volterra integral equation of the first kind:

$$\begin{aligned} \frac{1}{5}(\sin x - 2\cos x + 2e^2x) + \frac{1}{2}e^x(1 - e^{-1}(\cos 1 + \sin 1)) &= \int_0^x \cos(x-t)e^{2u(t)}dt \\ &+ \int_0^1 e^{x-t}\sin u(t)dt. \end{aligned}$$

Table 2 shows our results.

x	$m = 50, N = 10$	$e_{50,10}$	$m = 100, N = 10$	$e_{100,10}$	$exact solution$
0.00	0.0076	7.60×10^{-3}	0.0039	3.9×10^{-3}	0.0
0.1	0.112	1.12×10^{-2}	0.1041	4.1×10^{-3}	0.1
0.2	0.2084	8.4×10^{-3}	0.2043	4.3×10^{-3}	0.2
0.3	0.3113	1.13×10^{-2}	0.3044	4.4×10^{-3}	0.3
0.4	0.4089	8.90×10^{-3}	0.4045	4.5×10^{-3}	0.4
0.5	0.5108	1.08×10^{-2}	0.5046	4.6×10^{-3}	0.5
0.6	0.6093	9.3×10^{-3}	0.6047	4.7×10^{-3}	0.6
0.7	0.7106	1.06×10^{-2}	0.7047	4.7×10^{-3}	0.7
0.8	0.8095	9.5×10^{-3}	0.8048	4.8×10^{-3}	0.8
0.9	0.9103	1.03×10^{-2}	0.9048	4.8×10^{-3}	0.9

Table 2: Results of example 2

Example 5.3. Consider following equation with the exact solution $u(x) = x^2 - 1$:

$$u^{(3)}(x) + u'(x) = \frac{-x^5}{5} + \frac{2x^3}{3} + \frac{5x^2}{6} - \frac{113}{105}x - 1 + \int_0^x u^2(t)dt + \int_0^1 xt(x+t)u^2(t)dt$$

With the initial conditions: $u(0) = -1, u'(0) = 0, u''(0) = 2$.
Table 3 shows our results.

x	$m = 50, N = 10$	$e_{50,10}$	$m = 100, N = 10$	$e_{100,10}$	$exact solution$
0.00	-0.9998	2.00×10^{-4}	-1.00	0	-1
0.1	-0.9878	2.20×10^{-3}	-0.989	1.0×10^{-3}	-0.99
0.2	-0.9858	4.2×10^{-3}	-0.958	2.0×10^{-3}	-0.96
0.3	-0.9038	6.20×10^{-3}	-0.907	3.0×10^{-3}	-0.91
0.4	-0.8318	8.20×10^{-3}	-0.836	4.0×10^{-3}	-0.84
0.5	-0.7398	1.02×10^{-2}	-0.745	5.0×10^{-3}	-0.75
0.6	-0.6278	1.22×10^{-2}	-0.634	6.0×10^{-3}	-0.64
0.7	-0.4958	1.42×10^{-2}	-0.503	7.0×10^{-3}	-0.51
0.8	-0.3438	1.62×10^{-2}	-0.352	8.0×10^{-3}	-0.36
0.9	-0.1718	1.82×10^{-2}	-0.181	9.0×10^{-3}	-0.19

Table 3: Results of example 3

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