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Toeplitzness of weighted composition operators

M. Salehi Sarvestani

ABSTRACT. For a bounded analytic map ψ on the unit disk $\mathbb{D}$ and analytic self-map φ of $\mathbb{D}$, a weighted composition operator $C_{\psi,\varphi}$ on the Hardy space $H^2 = H^2(\mathbb{D})$ is defined by $C_{\psi,\varphi}f = \psi.f\circ\varphi$. In this paper, we study the asymptotically Toeplitzness of weighted composition operators and their adjoints in different topology on $B(H^2)$.

1. Introduction and Preliminaries

The Hardy space $H^2 = H^2(\mathbb{D})$ is a Hilbert space consisting of all holomorphic functions on disk $\mathbb{D}$ whose Fourier coefficients are square summable. The Hardy space can be consider as a closed subspace of $L^2(\partial\mathbb{D})$ (associated with normalized arc-length measure m on the unit circle $\partial\mathbb{D}$), since for each $f \in H^2$, by Fatou's theorem, the radial limit $f(e^{i\theta}) := \lim_{r\longrightarrow 1^-} f(re^{i\theta})$ exists a.e. with respect to arc-length measure on the unit circle $\partial\mathbb{D}$. Hence H^2 is isometrically isomorphic to a closed subspace of $L^2(\partial\mathbb{D})$.

An analytic self-map φ of the unit disk $\mathbb{D}$ (i.e. $\varphi(\mathbb{D}) \subseteq \mathbb{D}$) induces the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, where f is analytic on $\mathbb{D}$. It is well-know that the restriction of a composition operator to the Hardy space H^2 is a bounded operator on H^2 (for example see [**4**, pp.16].

For a function $\psi \in L^\infty(\partial\mathbb{D})$ (associated with normalized arc-lenght measure on $\partial\mathbb{D}$) the Toeplitz operator $T_\psi \in B(H^2)$ (the set of all bounded linear operators on H^2) defined by $T_\psi f = P(\psi f)$, where P is the orthonormal projection of $L^2(\partial\mathbb{D})$ onto H^2. In the case that ψ is a bounded analytic function on the unit disk, T_ψ is the multiplication operator on H^2, since $\psi H^2 \subseteq H^2$. If $\psi(z) \equiv z$, one obtains the forward shift operator $S = T_z$ and its adjoint, $S^* = T_{\overline{z}}$, the backward shift operator. It is well-know that an operator $T \in B(H^2)$ is a Toeplitz operator if and only if $S^*TS = T$. Barria and Halmos in [**1**] introduce asymptotic Toeplitz operators and define an operator $T \in B(H^2)$ is strongly asymptotically Toeplitz if the sequence

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$\{S^{*n}TS^n\}$ converges in strong operator toplogy on $B(H^2)$. This sense of asymptotic Toeplitzness has been extended to other usual topology on $B(H^2)$ by Feintuch [2]. Indeed an operator $T \in B(H^2)$ is called uniformly (weakly) asymptotic Toeplitz if the sequence $\{S^{*n}TS^n\}$ converges in norm (weak) topology on $B(H^2)$. In all cases the limit is a Toeplitz operator that its symbol is called the asymptotic symbol of T.

The space of all functions that are analytic and bounded on $\mathbb{D}$ denote by H^∞. For analytic self-map φ of $\mathbb{D}$ and a function $\psi \in H^\infty$, the bounded operator $C_{\psi,\varphi} : H^2 \longrightarrow H^2$ given by

$$C_{\psi,\varphi} f = \psi \cdot (f \circ \varphi) \ f \in H^2$$

is called a weighted composition operator. It is clear that $C_{\psi,\varphi} = T_\psi C_\varphi$.
The idea of this work come back to the paper [3]. The authors in [3] showed though non-trivial composition operators are not Toeplitz but a large classes of these operators are asymptotically closed to Toeplitz operators. They also proved a composition operator is uniformly asymptotic Toeplitz if and only if it is compact or identity. In the case that $\varphi(0) = 0$, they show all composition operators C_φ, except those induced by rotations, are weakly asymptotically Toeplitz. In this article, we try to extend some of these results to weighed composition operators.

2. Toeplitzness of weighted composition operators

Proposition 2.1. *The weighted composition operator $C_{\psi,\varphi}$ is Toeplitz provided that $C_{\psi,\varphi} = T_\psi$.*

Proof. If $\psi \equiv 0$, then it is trivial. Let ψ be non-zero. The matrix representation of $C_{\psi,\varphi}$ relative to the orthonormal basis of monomials $\{z^n : n \geq 0\}$ is of the form

$$\begin{bmatrix} \widehat{\psi}(0) & \widehat{\psi\varphi}(0) & \widehat{\psi\varphi^2}(0) & \cdots \\ \widehat{\psi}(1) & \widehat{\psi\varphi}(1) & \widehat{\psi\varphi^2}(1) & \cdots \\ \widehat{\psi}(2) & \widehat{\psi\varphi}(2) & \widehat{\psi\varphi^2}(2) & \cdots \\ \widehat{\psi}(3) & \widehat{\psi\varphi}(3) & \widehat{\psi\varphi^2}(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where "$\widehat{\ }$" denotes Fourier coefficient. If the weighted composition operator $C_{\psi,\varphi}$ is Toeplitz, then it has constant diagonals. Hence we have the following equations

$$\widehat{\psi}(n) = \widehat{\psi\varphi}(n+1), \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

and

$$\widehat{\psi\varphi}(n) = \widehat{\psi\varphi^2}(n+1), \qquad (n = 0, 1, 2, \ldots). \tag{3}$$

By equations (2) and (3) we show that φ is the identity (therefore $C_{\psi,\varphi}$ be the Toeplitz operator T_ψ). We have three cases:

Case I: Let $\widehat{\varphi}(0)=0$. If ψ is not zero function, then consider $\widehat{\psi}(m)$ be the first non-zero Fourier coefficient of ψ for some non-negative integer number m. Thus, by (2) we have $\widehat{\psi}(m)=\widehat{\psi\varphi}(m+1)=\widehat{\psi}(m)\widehat{\varphi}(1)$. Hence $\widehat{\varphi}(1)=1$ and so the equation $\widehat{\psi}(m+1)=\widehat{\psi}(m)\widehat{\psi}(2)+\widehat{\psi}(m+1)\widehat{\varphi}(1)$ implies that $\widehat{\varphi}(2)=0$. Proceeding by induction, assume that $\widehat{\varphi}(i)=0$, for all $2<i\leq n$ and some positive integer number n. Therefore, (2) implies that $\widehat{\psi}(m+n)=\widehat{\psi}(m)\widehat{\varphi}(n+1)+\widehat{\psi}(m+n)\widehat{\varphi}(1)$. Hence $\widehat{\varphi}(n+1)=0$. Therefore $\widehat{\varphi}(n)=0$ for $n>1$, by assumption. Hence $\varphi\equiv z$ and $C_{\varphi,\psi}=T_{\psi}$.

Case II: Let $\widehat{\varphi}(0)=0$ and $\widehat{\psi}(0)=0$. Then, equation (2) for $n=0$ implies $\widehat{\psi}(1)=0$. Suppose $\widehat{\psi}(i)=0$ for $i=2,3,\ldots,n$ where n is a intrger number bigger than 1. Therefore, by (2), $\widehat{\psi}(n)=\Sigma_{i=0}^{n+1}\widehat{\psi}(n-i+1)\widehat{\varphi}(i)=\widehat{\varphi}(0)\widehat{\psi}(n+1)$. So by induction $\psi\equiv 0$ and thus $C_{\psi,\varphi}=T_{\psi}$.

Case III: Let $\widehat{\varphi}(0)\neq 0$ and $\widehat{\psi}(0)\neq 0$. By equation (3) for $n=0$

$$\begin{aligned}\widehat{\psi}(0)\widehat{\varphi}(0)&=\widehat{\psi}(0)\widehat{\varphi^2}(1)+\widehat{\psi}(1)\widehat{\varphi^2}(0)\\&=2\widehat{\psi}(0)\widehat{\varphi}(0)\widehat{\varphi}(1)+\widehat{\psi}(1)\widehat{\varphi}(0)^2.\end{aligned}$$

Therefore

$$\widehat{\psi}(0)=2\widehat{\psi}(0)\widehat{\varphi}(1)+\widehat{\psi}(1)\widehat{\varphi}(0), \tag{4}$$

when $\widehat{\varphi(0)}\neq 0$. On the other hand, (2) implies

$$\widehat{\psi}(0)=\widehat{\psi}(0)\widehat{\varphi}(1)+\widehat{\psi}(1)\widehat{\varphi}(0) \tag{5}$$

The assumption $\widehat{\psi}(0)\neq 0$ and equations (4) and (5) imply $\widehat{\varphi}(1)=0$. Let now $\widehat{\varphi}(i)=0$ for $i=0,1,\ldots,n-1$, where n is a integer number bigger than 1. We show that $\widehat{\varphi}(n)=0$. We see that equation (3) for $n-1$ implies $\widehat{\psi\varphi}(n-1)=\widehat{\psi\varphi^2}(n)$ and so by the assumption

$$\begin{aligned}\widehat{\psi}(n-1)\widehat{\varphi}(0)&=\widehat{\psi}(0)\widehat{\varphi^2}(n)+\widehat{\psi}(n)\widehat{\varphi^2}(0)\\&=2\widehat{\psi}(0)\widehat{\varphi}(0)\widehat{\varphi}(n)+\widehat{\psi}(n)\widehat{\varphi}(0)^2.\end{aligned}$$

Therefore

$$\widehat{\psi}(n-1)=2\widehat{\psi}(0)\widehat{\varphi}(n)+\widehat{\psi}(n)\varphi(0), \tag{6}$$

since $\widehat{\varphi}\neq 0$. Moreover, by (2) we have

$$\widehat{\psi}(n-1)=\widehat{\psi\varphi}(n)=\widehat{\psi}(0)\widehat{\varphi}(n)+\widehat{\psi}(n)\widehat{\varphi}(0). \tag{7}$$

Hence, equations (6) and (7) show that $\widehat{\varphi}(n)=0$, since $\widehat{\psi}(0)\neq 0$. Thus by induction φ is the non-zero constant function $\varphi(z)=\widehat{\varphi}(0)$ and so equation (2) shows

$$\widehat{\psi}(n)=\widehat{\psi}(n+1)\widehat{\varphi}(0), \qquad (n=0,1,2,\ldots)$$

and thus

$$\widehat{\psi}(n)=\frac{\widehat{\psi}(0)}{\widehat{\varphi}(0)^n}, \qquad (n=0,1,2,\ldots).$$

Since $|\widehat{\varphi}(0)| < 1$, $\psi(\widehat{\varphi}(0)) = \Sigma_{n=0}^{\infty} \frac{\widehat{\psi}(0)}{\widehat{\varphi}(0)^n} \widehat{\varphi}(0)^n = \Sigma_{n=0}^{\infty} \widehat{\psi}(0)$. It shows $\widehat{\psi}(0) = 0$ and this is a contradiction. □

The reproducing kernel for $a \in \mathbb{D}$ is the function K_a in H^2 that defined by

$$K_a(z) = \frac{1}{1 - \overline{a}z}.$$

The H^2-norm of K_a is $\frac{1}{\sqrt{1-|a|^2}}$ and $k_a = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$ is called the normalized reproducing kernel. It is well know that for each $f \in H^2$, the values of f on the unit disk are reproduced by the reproducing functions in the following sense:

$$f(a) = \langle f, K_a \rangle, \ (a \in \mathbb{D}).$$

Another property of the reproducing kernels $\{K_a : a \in \mathbb{D}\}$ is that they are linear span dense in H^2. If $\{f_n\}$ is a sequence in H^2 which converges weakly to zero. Then it is convergent point-wise to zero on the unit disk, since

$$f_n(a) = \langle f_n, K_a \rangle \longrightarrow a$$

for every $a \in \mathbb{D}$. The converse is correct if the sequence $\{f_n\}$ is bounded in H^2.

Proposition 2.2. *Let $\{f_n\}$ be a bounded sequence in H^2. If $\{f_n\}$ converges point-wise on $\mathbb{D}$, then it is convergence weakly.*

Proof. If the sequence $\{f_n\}$ converges point-wise to zero on $\mathbb{D}$, then for each vector g in the linear span of $\{K_a : a \in \mathbb{D}\}$

$$\langle f_n, g \rangle \longrightarrow 0$$

as $n \longrightarrow \infty$. Let $f \in H^2$ and $\varepsilon > 0$ be arbitrary. By the property of reproducing kernels for some vector g_0 in the linear span $\{K_a : a \in \mathbb{D}\}$

$$\|f - g_0\| < \frac{\varepsilon}{2M}$$

where is a positive number that $\|f_n\| \leq M$ for each positive integer number n. Moreover there is a positive integer number N such that if $n \geq N$ that $|\langle f, g_0 \rangle| < \varepsilon/2$. Hence for each $n \geq N$

$$\begin{aligned} |\langle f_n, f \rangle| &\leq |\langle f_n, f - g_0 \rangle| + |\langle f_n, g_0 \rangle| \\ &\leq \|f_n\| \|f - g_0\| + \varepsilon/2 \\ &< M\varepsilon/2M + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

By using Feintuch characterization of uniformly asymptotically Toeplitz operators [2], the authors in [3] show that a composition operator is uniformly asymptotically Toeplitz if it is either compact or the identity. As follows we use their method to show that a weighted composition operator is uniformly asymptotically Toeplitz if it is Toeplitz or compact.

Theorem 2.3. *A weighted composition operator $C_{\psi,\varphi}$ on H^2 is uniformly asymptotically Toeplitz if and only if it is Toeplitz or compact.*

Proof. Let $C_{\psi,\varphi}$ be uniformly asymptotically Toeplitz. Therefore $C_{\psi,\varphi} = T_g + C$ for some $g \in L^\infty(\partial\mathbb{D})$ and compact operator C. We show that if φ is not the identity and $C_{\psi,\varphi}$ is not compact then $C = C_{\psi,\varphi}$ is not compact. Equivalently we show $C^* = C^*_{\psi,\varphi} - T^*_g$ is not compact. If K_a is the reproducing kernel for the point $a \in \mathbb{D}$ and f be a vector in H^2 then

$$\begin{aligned}
\langle C^*_{\psi,\varphi} K_a, f\rangle &= \langle (T_\varphi C_\varphi)^* K_a, f\rangle \\
&= \langle K_a, T_\psi C_\varphi f\rangle \\
&= \langle K_a, \psi \cdot f \circ \varphi\rangle \\
&= \overline{\psi(a) f(\varphi(a))} \\
&= \overline{\psi(a)} \langle K_{\varphi(a)}, f\rangle.
\end{aligned}$$

Therefore $C^*_{\psi,\varphi} K_a = \overline{\psi(a)} K_{\varphi(a)}$. By Proposition 2.2, $k_a = \dfrac{K_a}{\|K_a\|} = \sqrt{1-|a|^2} K_a$ converges weakly to zero, so to show C^* is not compact, it is enough to prove that the sequence $\{\|C^* k_a\|\}$ does not converge to zero as $|a| \longrightarrow 1^-$. For this we show the sequence $\{\langle C^* k_a, k_a\rangle\}$ is not convergent to zero.

$$\begin{aligned}
\langle C^* k_a, k_a\rangle &= \langle C^*_{\psi,\varphi} k_a, k_a\rangle - \langle T^*_g k_a, k_a\rangle \\
&= (1-|a|^2)\langle C^*_{\psi,\varphi} K_a, K_a\rangle - \langle P(\overline{g} k_a), k_a\rangle \\
&= (1-|a|^2)\langle \overline{\psi(a)} K_{\varphi(a)}, K_a\rangle - \langle P(\overline{g} k_a), k_a\rangle \\
&= (1-|a|^2)\overline{\psi(a)} K_{\varphi(a)}(a) - \langle P(\overline{g} k_a), k_a\rangle \\
&= \frac{1-|a|^2}{1-\overline{\varphi(a)}a}\overline{\psi(a)} - \langle P(\overline{g} k_a), k_a\rangle.
\end{aligned}$$

By the assumption g is not a.e zero on $\partial\mathbb{D}$, so according the method that be used in [3, Theorem 1.1] there is $\zeta \in \partial\mathbb{D}$ such that $\overline{g}(\zeta) \neq 0$, $\varphi(\zeta) \neq \zeta$ and

$$\langle \overline{g} k_{r\zeta}, k_{r\zeta}\rangle \longrightarrow \overline{g}(\zeta) \neq 0,$$

as $r \longrightarrow 1^-$. Moreover since ψ is bounded and $\varphi(\zeta) \neq \zeta$,

$$\frac{1-|r\zeta|^2}{1-\overline{\varphi(r\zeta)}r\zeta}\psi(r\zeta) \longrightarrow \frac{0}{1-\overline{\varphi(\zeta)}\zeta} = 0,$$

as $r \longrightarrow 1^-$. Hence the sequence $\{\langle C^* k_a, k_a\rangle\}$ is not convergent to zero as $|a| \longrightarrow 1^-$. □

3. Strongly asymptotically Toeplitzness of weighted composition operators and their adjoints

Proposition 3.1. *If $|\varphi| < 1$ on $\partial\mathbb{D}$, then $C_{\psi,\varphi}$ is strongly asymptotically Toeplitz with zero symbol.*

Proof. Let f be a vector in H^2. Then

$$\begin{aligned}\|S^{*n}C_{\psi,\varphi}S^n f\| &\leq \|T_\psi C_\varphi S^n f\| \\ &\leq \|T_\psi\|\|\varphi^n \cdot (f\circ\varphi)\| \\ &\leq \|T_\psi\|\|C_\varphi f\| \left\{\int_{\partial\mathbb{D}} |\varphi|^{2n} dm\right\}^{1/2}.\end{aligned}$$

Since $|\varphi| < 1$ on $\partial\mathbb{D}$, by Dominated Convergence Theorem $\int_{\partial\mathbb{D}} |\varphi|^{2n} dm$ converges to zero as $n \longrightarrow \infty$. Therefore $C_{\psi,\varphi}$ is strongly asymptotically Toeplitz. □

Proposition 3.2. *Let $\varphi \neq z$, $\varphi(0) = 0$ and $\psi \neq 0$. If $C_{\psi,\varphi}$ is strongly asymptotically Toeplitz, then $|\varphi| < 1$ a.e on $\partial\mathbb{D}$.*

Proof. If $C_{\psi,\varphi}$ is strongly asymptotically Toeplitz and φ is not identity and fixes the origin, then the asymptotic symbol is zero. Therefore

$$\lim_{n\to\infty} \|S^{*n}C_{\psi,\varphi}S^n 1\| = 0.$$

Set $E = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$ and put $\varphi = z\varphi_1$ for some holomorphic function on $\mathbb{D}$. For all non-negative integer n:

$$\begin{aligned}\|S^{*n}C_{\psi,\varphi}S^n 1\|^2 &= \int_{\partial\mathbb{D}} |S^{*n}T_\psi\varphi^n|^2 dm \\ &= \int_{\partial\mathbb{D}} |S^{*n}\psi z^n \varphi_1^n|^2 dm \\ &= \int_{\partial\mathbb{D}} |\psi|^2|\varphi_1|^{2n} dm \\ &= \int_E |\psi|^2 dm + \int_{\partial\mathbb{D}\setminus E} |\psi|^2|\varphi_1|^{2n} dm.\end{aligned}$$

By Dominated Convergence Theorem the sequence $\{\int_{\partial\mathbb{D}\setminus E} |\psi|^2|\varphi_1|^{2n} dm\}$ converges to zero as $n \longrightarrow \infty$. So $\int_E |\psi|^2 dm = 0$. Therefore if $m(E) \neq 0$, then $m(\{e^{i\theta} : \psi(e^{i\theta}) = 0\}) > 0$ and by F. M. Riesz Theorem $\psi = 0$ on $\mathbb{D}$, so $m(E) = 0$ and $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$. □

Nazarov and Shapiro in [3] show if the non-rotation φ fixes the origin, then C_φ^* is SAT with zero symbol. Similarly and with the same method we extend the result to the weighted composition operators. Let $A(\mathbb{D})$ be the collection of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ with continues extention on $\partial\mathbb{D}$ (i. e. $A(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$).

Theorem 3.3. *If $\varphi(0) = 0$, φ is not a rotation and $\psi \in A(\mathbb{D})$, then $C^*_{\psi,\varphi}$ is strongly asymptotically Toeplitz with zero symbol.*

Proof. The closed linear span of the reproducing kernels $\{K_a : a \in \mathbb{D}\}$ is H^2 and the $\{S^{*n}C^*_{\psi,\varphi}S^n\}$ is uniformly bounded, so it is enough to show that

$$\lim_{n\to\infty} \|S^{*n}C^*_{\psi,\varphi}S^nK_a\|$$

for $a \in \mathbb{D}$. First we prove $\lim_{n\to\infty} \|S^{*n}C^*_{\psi,\varphi}S^nK_a\|$ for $\psi = z^m$ such that m is a negative integer number.

$$T_{z^m}S^nK_a = P(z^m z^n K_a) = P(\Sigma_{i=0}^{\infty}\overline{a}^i\overline{z}^{(i+n+m)})$$

for $n + m \geq 0$. So

$$T_{z^m}S^nK_a = \frac{1}{\overline{a}^{(n+m)}}[K_a - P_{n+m-1}(z)]$$

where $P_{n+m-1}(z) = \Sigma_{k=0}^{n+m-1}(\overline{a}z)^k$.
Since $\varphi(0) = 0$, then C^*_φ has upper triangular matrix with respect to the orthonormal basis $\{z^n\}_{n=0}^{\infty}$ for H^2. Hence $C^*_\varphi P_{n+m-1}$ is a polynomial of degree less than $n + m$. Therefore $S^{*n}C^*_\varphi P_{n+m-1} = 0$ and then

$$S^{*n}C^*_\varphi T_{z^m}S^nK_a = \frac{1}{\overline{a}^{(n+m)}}S^{*n}K_{\varphi(a)}.$$

So

$$\|S^{*n}C^*_\varphi T_{z^m}S^nK_a\| = |\frac{1}{a^{n+m}}||\varphi(a)|^n\|K_{\varphi(a)}\| = |a^{-m}||\varphi(a)/a|^n\|K_{\varphi(a)}\|.$$

Since $\varphi(0) = 0$ and φ is not a rotation, the Schwarz Lemma says $|\varphi(a)| < |a|$. Hence $\|S^{*n}C^*_{z^m,\varphi}T_{z^m}S^nK_a\| \to 0$. Hence by linearity of the map $\phi \mapsto T_\phi$, if Q is a trigonometric polynomial then

$$\lim_{n\to\infty} \|S^{*n}C^*_{z^m,\varphi}T_{z^m}S^nK_a\| = 0,$$

for $a \in \mathbb{D}$. Let f is a non-zero vector in H^2. Trigonometric polynomials are dense in $C(\partial\mathbb{D})$, so by continuity of ψ and Stone Wirestress Theorem there is a trigonometric polynomial Q such that $\|\psi - Q\| < \|C^*_\varphi\|\|f\|\epsilon/2$. Moreover there is a positive integer number N such that for $n \geq N$, $\|S^{*n}C^*_\varphi T_{\overline{Q}}S^nf\| < \varepsilon/2$. If $n \leq N$, then

$$\begin{aligned}
\|S^{*n}C^*_\varphi T_{\overline{\psi}}S^nf\| &\leq \|S^{*n}C^*_\varphi(T_{\overline{\psi}} - T_{\overline{Q}})S^nf\| + \|S^{*n}C^*_\varphi T_{\overline{Q}}S^nf\| \\
&\leq \|C^*_\varphi\|\|f\|\|\psi - Q\| + \|S^{*n}C^*_\varphi T_{\overline{Q}}S^nf\| \\
&< \varepsilon.
\end{aligned}$$

Therefore, $C^*_{\psi,\varphi}$ is strongly asymptotically Toeplitz. □

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Department of mathematics, Savadkoh Branch, Islamic Azad university, Savadkoh, Iran.

Email address: m.salehisarvestani@gmail.com