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Non-stabilities of mixed type Euler-Lagrange k-cubic-quartic functional equation in various normed spaces

J. M. Rassias, M. Arunkumar*, and E. Sathya

Abstract. In this paper, we introduce and examine the generalized Ulam-Hyers stability of fixed Euler-Lagrange k-Cubic-Quartic functional Equation

$$\begin{aligned} &f(x+ky)+f(kx+y)+f(x-ky)+f(y-kx)\\ &\qquad = k^2\left[2f(x+y)+f(x-y)+f(y-x)\right]+2(k^4-1)[f(x)+f(y)]\\ &\qquad\qquad + \frac{k^2}{4}(k^2-1)[f(2x)+f(2y)] \end{aligned}$$

where k is a real number with $k \neq 0, \pm 1$ in various Banach spaces with the help of two different methods.

1. Introduction

A fundamental question in the theory of functional equations is as follows: *When is it true that a function that approximately satisfies a functional equation must be close to an exact solution of the equation?*

If the problem allows a solution, we say that the equation is stable. The first stability problem concerning of group homomorphisms was introduced by Ulam [62] in 1940. The famous Ulam stability problem was partially solved by Hyers [34] for the linear functional equation of Banach spaces. Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [56], [49] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [30] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The terminology

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*Corresponding author.

Hyers-Ulam-Rassias stability originates from these historical backgrounds and this terminology is also applied to the cases of other functional equations.

Cadariu and Radu [24] applied the fixed point method for investigation of the Jensen functional equation. They could present a short and a simple proof (different from the direct method initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of the Jensen functional equation, quadratic functional equation [25] and additive functional equation [26]. Their methods are a powerful tool for studying the stability of several functional equations.

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (c.f. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 29, 32, 36, 45, 47, 48, 52, 53, 54, 55, 64, 65, 66]) and references therein.

The solution and stability of following cubic-quartic functional equations

$$\begin{aligned} &f(x+kx)+f(x-ky)\\ &\ = k^2\left\{f(x+y)+f(x-y)\right\}-2(k^2-1)f(x)-2k^2(k^2-1)f(y)\\ &\qquad +\frac{k^2(k^2-1)}{4}f(2y); k\neq 0,\pm 1,\\ &f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y) \end{aligned}$$

and

$$\begin{aligned} &f(2x+y)+f(2x-y)\\ &\qquad = 3f(x+y)+f(-x-y)+3f(x-y)+f(y-x)\\ &\qquad\quad +18f(x)+6f(-x)-3f(y)-3f(-y) \end{aligned}$$

were introduced and investigated by [21, 31, 46]

In this paper, we introduce and examine the generalized Ulam-Hyers stability of Mixed Euler-Lagrange $k-$Cubic-Quartic functional Equation

$$\begin{aligned} &f(x+ky)+f(kx+y)+f(x-ky)+f(y-kx)\\ &= k^2\left[2f(x+y)+f(x-y)+f(y-x)\right]-2(k^4-1)[f(x)+f(y)]\\ &\qquad +\frac{k^2}{4}(k^2-1)[f(2x)+f(2y)] \end{aligned} \tag{1}$$

where k is a real number with $k\neq 0,\pm 1$ in various Banach spaces with the help of two different methods.

2. Solution of the functional equation

In this section, we provide the solution of the functional equation (1) by considering A and B are real vector spaces.

Theorem 2.1. *Let an odd $f: A \longrightarrow B$ be a mapping satisfying (1), for all $x,y\in A$, then f is cubic.*

Proof. Changing (x, y) by $(0,0)$ in (1), we arrive $f(0) = 0$. Replacing y by 0 in (1), we get

$$\begin{aligned} f(x)+f(kx)+f(x)+f(-kx) = k^2\left[2f(x)+f(x)+f(-x)\right] - 2(k^4-1)f(x) \\ + \frac{k^2}{4}(k^2-1)f(2x) \end{aligned} \tag{2}$$

for all $x \in A$. Using oddness of f in (2), we obtain

$$2f(x) = 2k^2 f(x) - 2(k^4-1)f(x) + \frac{k^2}{4}(k^2-1)f(2x) \tag{3}$$

for all $x \in A$. It follows from (3) that

$$\frac{k^2}{4}(k^2-1)f(2x) = \left(2 - 2k^2 + 2(k^4-1)\right) f(x);$$

or

$$\frac{k^2}{4}(k^2-1)f(2x) = \left(-2k^2 + 2k^4\right) f(x);$$

or

$$k^2(k^2-1)f(2x) = 2^3 k^2\left(k^2-1\right) f(x) \tag{4}$$

for all $x \in A$. Since $k \neq 0, \pm 1$, the above equation yields

$$f(2x) = 2^3 f(x) \tag{5}$$

for all $x \in A$. Hence f is cubic. □

Theorem 2.2. *Assume that $f : A \longrightarrow B$ is an even mapping satisfying (1), for all $x, y \in A$, then f is quartic.*

Proof. Changing (x, y) by $(0,0)$ in (1), we arrive $f(0) = 0$. Replacing y by 0 in in (1), we get

$$\begin{aligned} f(x)+f(kx)+f(x)+f(-kx) = k^2\left[2f(x)+f(x)+f(-x)\right] - 2(k^4-1)f(x) \\ + \frac{k^2}{4}(k^2-1)f(2x) \end{aligned} \tag{6}$$

for all $x \in A$. Using evenness of f in (6), we obtain

$$2f(x) + 2f(kx) = 4k^2 f(x) - 2(k^4-1)f(x) + \frac{k^2}{4}(k^2-1)f(2x) \tag{7}$$

for all $x \in A$. Using $f(2x) = 2^4 f(x)$ in (7), we have

$$2f(x) + 2f(kx) = 4k^2 f(x) - 2(k^4-1)f(x) + 4k^2(k^2-1)f(x) \tag{8}$$

for all $x \in A$. It follows from (4) that

$$2f(kx) = \left(4k^2 - 2(k^4-1) + 4k^2(k^2-1) - 2\right) f(x)$$

or

$$2f(kx) = 2k^4 f(x) \tag{9}$$

for all $x \in A$. It follows from above equation that

$$f(kx) = k^4 f(x) \tag{10}$$

for all $x \in A$. Hence f is quartic. □

Hereafter, throughout this paper let us assume $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ by

$$\begin{aligned} F_k(x,y) = {} & f(x+ky) + f(kx+y) + f(x-ky) + f(y-kx) \\ & - k^2\left[2f(x+y) + f(x-y) + f(y-x)\right] \\ & + 2(k^4-1)[f(x)+f(y)] - \frac{k^2}{4}(k^2-1)[f(2x)+f(2y)] \end{aligned}$$

where k is a real number with $k \neq 0, \pm 1$.

All the stability results are proved by Hyers and Radus Method by taking the function f is odd, even and odd-even cases.

3. Stability in Banach space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equation (1) in Banach space. To prove stability results, let us take $\mathcal{B}_1$ be a normed space and $\mathcal{B}_2$ be a Banach space.

3.1. Hyers Method.

Theorem 3.1. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality*

$$\|F_k(x,y)\| \leq L(x,y) \tag{11}$$

for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0,\infty)$ satisfies the condition

$$\lim_{N\to\infty} \frac{L(2^{NJ}x, 2^{NJ}y)}{2^{3NJ}} = 0 \tag{12}$$

for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C}(x) : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{C}(x) - f(x)\| \leq \frac{1}{\lambda} \sum_{M=\frac{1-J}{2}}^{\infty} \frac{L(2^{MJ}x, 0)}{2^{3MJ}} \tag{13}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$ and $\lambda = 2k^2(k^2-1)$. The mapping $\mathcal{C}$ is defined by

$$\mathcal{C}(x) = \lim_{N\to\infty} \frac{L(2^{NJ}x)}{2^{3NJ}} \tag{14}$$

for all $x \in \mathcal{B}_1$.

Proof. Changing y by 0 in (11), we reach

$$\Big\| f(x)+f(kx)+f(x)+f(-kx)-k^2\left[2f(x)+f(x)+f(-x)\right] \\ +2(k^4-1)f(x)-\frac{k^2}{4}(k^2-1)f(2x)\Big\| \leq L(x,0) \quad (15)$$

for all $x \in \mathcal{B}_1$. Using oddness of f in (15), we obtain

$$\left\|2f(x)-2k^2f(x)+2(k^4-1)f(x)-\frac{k^2}{4}(k^2-1)f(2x)\right\| \leq L(x,0); \textit{or}$$

$$\left\|-\frac{k^2}{4}(k^2-1)f(2x)+2k^2\left(k^2-1\right)f(x)\right\| \leq L(x,0) \quad (16)$$

for all $x \in \mathcal{B}_1$. It follows from (16) and $k \neq 0, \pm 1$,

$$\left\|\frac{f(2x)}{2^3}-f(x)\right\| \leq \frac{L(x,0)}{2k^2\left(k^2-1\right)} \quad (17)$$

for all $x \in \mathcal{B}_1$. Let $\lambda = 2k^2\left(k^2-1\right)$ in above inequality, we arrive

$$\left\|\frac{f(2x)}{2^3}-f(x)\right\| \leq \frac{L(x,0)}{\lambda} \quad (18)$$

for all $x \in \mathcal{B}_1$. Changing x by $2x$ and multiplying by $\frac{1}{2^3}$ in (18), we get

$$\left\|\frac{f(2^2x)}{2^6}-\frac{f(2x)}{2^3}\right\| \leq \frac{L(2x,0)}{\lambda\cdot 2^3} \quad (19)$$

for all $x \in \mathcal{B}_1$. Using triangle inequality on (18) and (19), we have

$$\begin{aligned}\left\|\frac{f(2^2x)}{2^6}-f(x)\right\| &= \left\|\frac{f(2^2x)}{2^6}-\frac{f(2x)}{2^3}+\frac{f(2x)}{2^3}-f(x)\right\| \\ &\leq \left\|\frac{f(2x)}{2^3}-f(x)\right\| + \left\|\frac{f(2^2x)}{2^6}-\frac{f(2x)}{2^3}\right\| \\ &\leq \frac{L(x,0)}{\lambda}+\frac{L(2x,0)}{\lambda\cdot 2^3} = \frac{1}{\lambda}\left[L(x,0)+\frac{L(2x,0)}{2^3}\right] \end{aligned} \quad (20)$$

for all $x \in \mathcal{B}_1$. Generalizing for a positive integer N, we obtain

$$\left\|\frac{f(2^Nx)}{2^{3N}}-f(x)\right\| \leq \frac{1}{\lambda}\sum_{M=0}^{N-1}\frac{L(2^Mx,0)}{2^{3M}} \quad (21)$$

for all $x \in \mathcal{B}_1$. Hence $\left\{\dfrac{f(2^N x)}{2^{3N}}\right\}$ is a Cauchy sequence and it converges to a point $\mathcal{C}(x) \in \mathcal{B}_2$. Indeed, replacing x by $2^P w$ and dividing by 2^{3P} in (21), we get

$$\begin{aligned}\left\|\frac{f(2^{N+P}x)}{2^{3(N+P)}} - \frac{f(2^P x)}{2^{3P}}\right\| &= \frac{1}{2^P}\left\|\frac{f(2^N \cdot 2^P x)}{2^{3N}} - f(2^P x)\right\| \\ &\leq \frac{1}{\lambda}\sum_{M=0}^{N-1}\frac{L(2^{M+P}x,0)}{2^{3(M+P)}} \\ &\longrightarrow 0 \quad as \quad P \quad \longrightarrow \quad \infty \end{aligned} \tag{22}$$

for all $x \in \mathcal{B}_1$. Thus, we define mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ by

$$\mathcal{C}(x) = \lim_{N\longrightarrow\infty}\frac{f(2^N x)}{2^{3N}}$$

for all $x \in \mathcal{B}_1$. Letting limit $N \to \infty$ in (21) and using the definition of $\mathcal{C}$, we get

$$\left\|\lim_{N\longrightarrow\infty}\frac{f(2^N x)}{2^{3N}} - f(x)\right\| \leq \frac{1}{\lambda}\sum_{M=0}^{\infty}\frac{L(2^M x,0)}{2^{3M}} \Longrightarrow \|\mathcal{C}(x) - f(x)\| \leq \frac{1}{\lambda}\sum_{M=0}^{\infty}\frac{L(2^M x,0)}{2^{3M}}$$

for all $x \in \mathcal{B}_1$. Thus (13) holds for $J = 1$. Now, to show that $\mathcal{C}$ satisfies (1), changing (x, y) by $(2^N x, 2^N y)$ and divided by 2^{3N} in (11), we reach

$$\frac{1}{2^{3N}}\left\|F_k(2^N x, 2^N y)\right\| \leq \frac{1}{2^{3N}}L(2^N x, 2^N y)$$

for all $x, y \in \mathcal{B}_1$. Approaching $N \to \infty$ and using the definition of $\mathcal{C}$, (12) in the above inequality, we can see that $\mathcal{C}$ satisfies the functional equation (1) for all $x \in \mathcal{B}_1$. In order to prove the existence of $\mathcal{C}$ is unique, assume that $\mathcal{C}'$ be another cubic mapping satisfying (1) and (13). Now,

$$\begin{aligned}\|\mathcal{C}(x) - \mathcal{C}'(x)\| &= \frac{1}{2^{3P}}\left\|\mathcal{C}(2^P x) - \mathcal{C}'(2^P x)\right\| \\ &= \frac{1}{2^{3P}}\left\|\mathcal{C}(2^P x) - f(2^P x) + f(2^P x) - \mathcal{C}'(2^P x)\right\| \\ &= \frac{1}{2^{3P}}\left\{\left\|\mathcal{C}(2^P x) - f(2^P x)\right\| + \left\|\mathcal{C}'(2^P x) - f(2^P x)\right\|\right\} \\ &\leq \frac{2}{\lambda}\sum_{M=0}^{\infty}\frac{L(2^{M+P}x,0)}{2^{3(M+P)}} \\ &\longrightarrow 0 \quad as \quad P \quad \longrightarrow \quad \infty\end{aligned}$$

for all $x \in \mathcal{B}_1$. This proves that $\mathcal{C}(x) = \mathcal{C}'(x)$, for all $x \in \mathcal{B}_1$. Thus, $\mathcal{C}(x)$ is unique. Hence, the theorem holds for $J = 1$.

Replacing x by $\dfrac{x}{2}$ in (18), we achieve

$$\left\|f(x) - 2^3 f\left(\frac{x}{2}\right)\right\| \leq \frac{2^3 L\left(\frac{x}{2},0\right)}{\lambda} \tag{23}$$

for all $x \in \mathcal{B}_1$. Again replacing x by $\dfrac{x}{2}$ and multiply by 2^3 in (23), we arrive

$$\left\| 2^3 f\left(\frac{x}{2}\right) - 2^6 f\left(\frac{x}{2^2}\right)\right\| \leq \frac{2^6 L\left(\frac{x}{2^2}, 0\right)}{\lambda} \tag{24}$$

for all $x \in \mathcal{B}_1$. Using triangle inequality on (23) and (24), we have

$$\begin{aligned} \left\| f(x) - 2^6 f\left(\frac{x}{2^2}\right)\right\| &\leq \left\| f(x) - 2^3 f\left(\frac{x}{2}\right)\right\| + \left\| 2^3 f\left(\frac{x}{2}\right) - 2^6 L\left(\frac{x}{2^2}\right)\right\| \\ &\leq \frac{2^3 L\left(\frac{x}{2}, 0\right)}{\lambda} + \frac{2^6 L\left(\frac{x}{2^2}, 0\right)}{\lambda} \\ &= \frac{1}{\lambda}\left[2^3 L\left(\frac{x}{2}, 0\right) + 2^6 L\left(\frac{x}{2^2}, 0\right)\right] \end{aligned} \tag{25}$$

for all $x \in \mathcal{B}_1$. Generalizing for a positive integer N, we obtain

$$\left\| f(x) - 2^{3N} f\left(\frac{x}{2^N}\right)\right\| \leq \frac{1}{\lambda}\sum_{M=1}^{N-1} 2^M L\left(\frac{x}{2^M}, 0\right) = \frac{1}{\lambda}\sum_{M=1}^{N-1} 2^M L\left(\frac{x}{2^M}, 0\right) \tag{26}$$

for all $x \in \mathcal{B}_1$. The rest of the proof is similar ideas to that of case $J = 1$. Thus the theorem is true for $J = -1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stabilities of (1).

Corollary 3.2. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality*

$$\|F_k(x, y)\| \leq \begin{cases} S; & \\ S\left(||x||^R + ||y||^R\right); & R \neq 3 \\ S\left(||x||^R ||y||^R + ||x||^{2R} + ||y||^{2R}\right); & 2R \neq 3 \end{cases} \tag{27}$$

for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{C}(x) - f(x)\| \leq \begin{cases} \dfrac{S}{\lambda|2^3 - 1|}; \\ \dfrac{S||x||^R}{\lambda|2^3 - 2^R|}; \\ \dfrac{S||x||^{2R}}{\lambda|2^3 - 2^{2R}|}; \end{cases} \tag{28}$$

for all $x \in \mathcal{B}_1$.

The following example is to illustrate that the functional equation (1) is not stable for $R = 3$ in Corollary 3.2.

Example 3.1. Let $L : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$L(x) = \begin{cases} Tx^3, & \text{if } |x| < 1 \\ T, & \text{otherwise} \end{cases}$$

where $T > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{M=0}^{\infty} \frac{L(2^M x)}{8^M} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\Big| f(x+ky) + f(kx+y) + f(x-ky) + f(y-kx) - k^2\left[2f(x+y) + f(x-y) + f(y-x)\right]$$
$$+2(k^4-1)[f(x)+f(y)] - \frac{k^2}{4}(k^2-1)[f(2x)+f(2y)]\Big|$$
$$\leq \frac{4T \times 8^2(9k^4-5k^2)}{7}\left(|x|^3+|y|^3\right) \tag{29}$$

for all $x, y \in \mathbb{R}$. Then there are not a cubic mapping such as $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $B > 0$ such that

$$|f(x) - \mathcal{C}(x)| \leq B|x|^3 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{30}$$

Proof. From the definition of $f(x)$, we have

$$|f(x)| \leq \sum_{M=0}^{\infty} \frac{|L(2^M x)|}{|8^M|} = \sum_{n=0}^{\infty} \frac{T}{8^M} = \frac{8T}{7}.$$

Thus, f is bounded. Now, we are going to prove that f satisfies (29).

If $x = y = 0$ then (29) is trivial. If $|x|^3 + |y|^3 \geq \frac{1}{8}$, then the left hand side of (29) is less than $\frac{4T(9k^4-5k^2)}{7}$. Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{8}$. Then there is a positive integer b such that

$$\frac{1}{8^{b+2}} \leq |x|^3 + |y|^3 < \frac{1}{8^{b+1}}, \tag{31}$$

so that $8^b|x|^3 < \frac{1}{8}$, $8^b|y|^3 < \frac{1}{8}$, and consequently

$$2^{b-1}(x+ky), 2^{b-1}(kx+y), 2^{b-1}(x-ky), 2^{b-1}(y-kx), 2^{b-1}(x+y),$$
$$2^{b-1}(x-y), 2^{b-1}(y-x), 2^{b-1}(x), 2^{b-1}(y), 2^{b-1}(2x), 2^{b-1}(2y) \in (-1,1).$$

Therefore for each $M = 0, 1, \ldots, b-1$, we have

$$2^M(x+ky), 2^M(kx+y), 2^M(x-ky), 2^M(y-kx), 2^M(x+y), 2^M(x-y) 2^M(y-x),$$
$$2^M(x), 2^M(y), 2^M(2x), 2^M(2y) \in (-1,1).$$

and

$$L(2^M(x+ky)) + L(2^M(kx+y)) + L(2^M(x-ky)) + L(2^M(y-kx))$$
$$-k^2\left[2L(2^M(x+y)) + L(2^M(x-y)) + L(2^M(y-x))\right]$$
$$+2(k^4-1))[L(2^M(x)) + L(2^M(y))] - \frac{k^2}{4}(k^2-1)[L(2^M(2x)) + L(2^M(2y))] = 0$$

for $M = 0, 1, \ldots, k-1$. From the definition of f and (31), we obtain that

$$\begin{aligned}
&\Big| f(x+ky) + f(kx+y) + f(x-ky) + f(y-kx) - k^2 \left[2f(x+y) + f(x-y) + f(y-x)\right] \\
&+ 2(k^4-1)[f(x)+f(y)] - \frac{k^2}{4}(k^2-1)[f(2x)+f(2y)]\Big| \\
&\leq \sum_{M=0}^{\infty} \frac{1}{8^M}\Big| L(2^M(x+ky)) + L(2^M(kx+y)) + L(2^M(x-ky)) + L(2^M(y-kx)) \\
&\quad - k^2 \left[2L(2^M(x+y)) + L(2^M(x-y)) + L(2^M(y-x))\right] \\
&\qquad + 2(k^4-1))[L(2^M(x)) + L(2^M(y))] - \frac{k^2}{4}(k^2-1)[L(2^M(2x)) + L(2^M(2y))]\Big| \\
&\leq \sum_{M=k}^{\infty} \frac{1}{8^M}\Big| L(2^M(x+ky)) + L(2^M(kx+y)) + L(2^M(x-ky)) + L(2^M(y-kx)) \\
&\quad - k^2 \left[2L(2^M(x+y)) + L(2^M(x-y)) + L(2^M(y-x))\right] \\
&\qquad + 2(k^4-1))[L(2^M(x)) + L(2^M(y))] - \frac{k^2}{4}(k^2-1)[L(2^M(2x)) + L(2^M(2y))]\Big| \\
&\leq \sum_{M=k}^{\infty} \frac{1}{8^M} \cdot \frac{9k^4-5k^2}{2} = \frac{8}{7} \cdot \frac{9k^4-5k^2}{2} \cdot \frac{1}{8^b} = \frac{4T \times 8^2(9k^4-5k^2)}{7}\left(|x|^3+|y|^3\right).
\end{aligned}$$

Thus f satisfies (29) for all $x, y \in \mathbb{R}$ with $0 < |x|^3 + |y|^3 < \frac{1}{8}$.

We claim that the cubic functional equation (1) is not stable for $R = 3$ in Corollary 3.2. Suppose on the contrary that there exist a cubic mapping $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $B > 0$ satisfying (30). Since f is bounded and continuous for all $x \in \mathbb{R}$, $\mathcal{C}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, $\mathcal{C}$ must have the form $\mathcal{C}(x) = cx^3$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f(x)| \leq (B + |c|)\, |x|^3. \tag{32}$$

But we can choose a positive integer q with $qT > B + |c|$.

If $x \in \left(0, \frac{1}{2^{q-1}}\right)$, then $2^M x \in (0,1)$ for all $M = 0, 1, \ldots, q-1$. For this x, we get

$$f(x) = \sum_{M=0}^{\infty} \frac{L(2^M x)}{8^M} \geq \sum_{M=0}^{q-1} \frac{T(2^M x)^3}{8^M} = qTx^3 > (B + |c|)\, x^3$$

which contradicts (32). Therefore the cubic functional equation (1) is not stable in sense of Ulam, Hyers and Rassias if $R = 3$, assumed in the inequality (28). □

Theorem 3.3. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality*

$$\|F_k(x,y)\| \leq L(x,y) \tag{33}$$

for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satifies the condition

$$\lim_{N\to\infty} \frac{L(k^{NJ}x, k^{NJ}y)}{k^{4NJ}} = 0 \tag{34}$$

for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{Q}(x) - f(x)\| \leq \frac{1}{2k^4} \sum_{M=\frac{1-J}{2}}^{\infty} \frac{L(k^{MJ}x, 0)}{k^{4MJ}} \tag{35}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$. The mapping $\mathcal{Q}$ is defined by

$$\mathcal{Q}(x) = \lim_{N\to\infty} \frac{L(k^{NJ}x)}{k^{4NJ}} \tag{36}$$

for all $x \in \mathcal{B}_1$.

Proof. Changing y by 0 in (33), we reach

$$\begin{aligned} \Big\| \, & f(x) + f(kx) + f(x) + f(-kx) - k^2\left[2f(x) + f(x) + f(-x)\right] \\ & +2(k^4-1)f(x) - \frac{k^2}{4}(k^2-1)f(2x)\Big\| \leq L(x, 0) \end{aligned} \tag{37}$$

for all $x \in \mathcal{B}_1$. Using evenness of f and $f(2x) = 2^4 f(x)$ in (37), we obtain

$$\begin{aligned} & \left\|2f(kx) + 2f(x) - 4k^2 f(x) + 2(k^4-1)f(x) - \frac{k^2}{4}(k^2-1)f(2x)\right\| \leq L(x,0), \\ & \left\|2f(kx) + 2f(x) - 4k^2 f(x) + 2(k^4-1)f(x) - 4\,k^2(k^2-1)f(x)\right\| \leq L(x,0), \\ & \left\|2f(kx) - 2k^4 f(x)\right\| \leq L(x,0), \end{aligned} \tag{38}$$

for all $x \in \mathcal{B}_1$. It follows from (38) and $k \neq 0, \pm 1$,

$$\left\|\frac{f(kx)}{k^4} - f(x)\right\| \leq \frac{L(x,0)}{2k^4} \tag{39}$$

for all $x \in \mathcal{B}_1$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 3.3 concerning the stabilities of (1).

Corollary 3.4. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality*

$$\|F_k(x,y)\| \leq \begin{cases} S; & \\ S\left(||x||^R + ||y||^R\right); & R \neq 4 \\ S\left(||x||^R ||y||^R + ||x||^{2R} + ||y||^{2R}\right); & R \neq 2 \end{cases} \tag{40}$$

for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{Q}(x) - f(x)\| \leq \begin{cases} \dfrac{S}{2|k^4-1|}; \\ \dfrac{S||x||^R}{2|k^4-k^R|}; \\ \dfrac{S||x||^{2R}}{2|k^4-k^{2R}|}; \end{cases} \tag{41}$$

for all $x \in \mathcal{B}_1$.

The following example is to illustrate that the functional equation (1) is not stable for $R = 4$ in Corollary 3.4.

Example 3.2. Let $L : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$L(x) = \begin{cases} Tx^4, & \text{if } |x| < 1 \\ T, & \text{otherwise} \end{cases}$$

where $T > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{M=0}^{\infty} \frac{L(k^M x)}{k^{4M}} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\Big| f(x+ky) + f(kx+y) + f(x-ky) + f(y-kx) - k^2\left[2f(x+y) + f(x-y) + f(y-x)\right] \\ +2(k^4-1)[f(x)+f(y)] - \frac{k^2}{4}(k^2-1)[f(2x)+f(2y)] \Big| \\ \leq \frac{k^4 T(9k^4 - 5k^2)}{2(k^4-1)} \left(|x|^4 + |y|^4\right) \tag{42}$$

for all $x, y \in \mathbb{R}$. Then there are not a quartic mapping $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ and a constant $B > 0$ such that

$$|f(x) - \mathcal{Q}(x)| \leq B|x|^4 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{43}$$

Theorem 3.5. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality*

$$\|F_k(x,y)\| \leq L(x,y) \tag{44}$$

for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satisfies the conditions (12) and (34), for all $x \in \mathcal{B}_1$. Then there are only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one *quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) such that*

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \frac{1}{2}\left\{\frac{1}{\lambda}\sum_{M=\frac{1-J}{2}}^{\infty}\left(\frac{L(2^{MJ}x,0)}{2^{3MJ}} + \frac{L(-2^{MJ}x,0)}{2^{3MJ}}\right)\right.$$
$$\left. + \frac{1}{2k^4}\sum_{M=\frac{1-J}{2}}^{\infty}\left(\frac{L(k^{MJ}x,0)}{k^{4MJ}} + \frac{L(-k^{MJ}x,0)}{k^{4MJ}}\right)\right\} \tag{45}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$. The mappings $\mathcal{C}$ and $\mathcal{Q}$ are defined in (14) and (36) respectively, for all $x \in \mathcal{B}_1$.

Proof. Assume that

$$f_c(x) = \frac{f(x) - f(-x)}{2},$$

for all $x \in \mathcal{B}_1$. It is easy to verify that

$$f_c(0) = 0 \qquad \text{and} \qquad f_c(-x) = -f_c(x),$$

for all $x \in \mathcal{B}_1$. By Theorem 3.1 and definition of f_c, we have

$$\|f_c(x) - \mathcal{C}(x)\| \leq \frac{1}{2}\cdot\frac{1}{\lambda}\sum_{M=\frac{1-J}{2}}^{\infty}\left(\frac{L(2^{MJ}x,0)}{2^{3MJ}} + \frac{L(-2^{MJ}x,0)}{2^{3MJ}}\right) \tag{46}$$

for all $x \in \mathcal{B}_1$. Also, assume that

$$f_q(x) = \frac{f(x) + f(-x)}{2},$$

for all $x \in \mathcal{B}_1$. It is easy to verify that

$$f_q(0) = 0 \qquad \textit{and} \qquad f_q(-x) = f_q(x),$$

for all $x \in \mathcal{B}_1$. By Theorem 3.3 and definition of f_q, we have

$$\|f_q(x) - \mathcal{Q}(x)\| \leq \frac{1}{2}\cdot\frac{1}{2k^4}\sum_{M=\frac{1-J}{2}}^{\infty}\left(\frac{L(k^{MJ}x,0)}{k^{4MJ}} + \frac{L(-k^{MJ}x,0)}{k^{4MJ}}\right) \tag{47}$$

for all $x \in \mathcal{B}_1$. Let us define

$$f(x) = f_c(x) + f_q(x) \tag{48}$$

for all $x \in \mathcal{B}_1$. Then by (46), (47) and (48), we arrive our result. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stabilities of (1).

Corollary 3.6. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality*

$$\|F_k(x,y)\| \leq \begin{cases} S; & \\ S\left(||x||^R + ||y||^R\right); & R \neq 3,4 \\ S\left(||x||^R||y||^R + ||x||^{2R} + ||y||^{2R}\right); & 2R \neq 3,4 \end{cases} \tag{49}$$

for all $x \in \mathcal{B}_1$. Then there are only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ such that satisfying the functional equation (1) and

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \begin{cases} \dfrac{S}{2}\left(\dfrac{1}{\lambda|2^3-1|} + \dfrac{1}{2|k^4-1|}\right); \\ \dfrac{S||x||^R}{2}\left(\dfrac{1}{\lambda|2^3-2^R|} + \dfrac{1}{2|k^4-k^R|}\right); \\ \dfrac{S||x||^{2R}}{2}\left(\dfrac{1}{\lambda|2^3-2^{2R}|} + \dfrac{1}{2|k^4-k^{2R}|}\right); \end{cases} \tag{50}$$

for all $x \in \mathcal{B}_1$.

3.2. Radu's method.

Theorem 3.7. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (11) for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{N\to\infty} \frac{L(\ell_I^N x, \ell_I^N y)}{\ell_I^{3N}} = 0 \tag{51}$$

where

$$\ell_I = \begin{cases} 2 & if \quad I = 0, \\ \frac{1}{2} & if \quad I = 1 \end{cases} \tag{52}$$

holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that for the function

$$L(x,0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right)$$

we have the following property

$$\frac{1}{\ell_I^3} L(\ell_I x, 0) = \mathcal{L} L(w,0), \tag{53}$$

for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{C}(x) - f(x)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(x,0) \tag{54}$$

for all $x \in \mathcal{B}_1$.

Proof. Consider the set

$$\mathcal{A} = \{f/f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2,\ f(0) = 0\}$$

and introduce the generalized metric $d : \mathcal{A} \times \mathcal{A} \to [0, \infty]$ as follows:

$$d(f, g) = \inf\{\omega \in (0, \infty) : \|f(x) - g(x)\| \leq \omega \ L(x, 0), x \in \mathcal{B}_1\}. \tag{55}$$

It is easy to show that $(\mathcal{A}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $U : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Uf(x) = \frac{1}{\ell_I^3} f_a(\ell_I x),$$

for all $x \in \mathcal{B}_1$. For given $f, f_a \in \mathcal{A}$ and $d(f, f_a) \in \omega$ that is

$$\|f(x) - f_a(x)\| \leq \omega \ L(x, 0), x \in \mathcal{B}_1.$$

So, we have

$$\begin{aligned} \|f(x) - f_a(x)\| &= \left\| \frac{1}{\ell_I^3} f(\ell_I x) - \frac{1}{\ell_I^3} f_a(\ell_I x) \right\| \\ &\leq \frac{\omega}{\ell_I^3} L(\ell_I x, 0) \\ &= \mathcal{L} \omega L(x, 0) \end{aligned}$$

for all $x \in \mathcal{B}_1$, that is,

$$d(Uf, Uf_a) \leq \mathcal{L} d(f, f_a), \qquad \forall \ f, f_a \in \mathcal{A}.$$

This implies U is a strictly contractive mapping on $\mathcal{A}$ with Lipschitz constant $\mathcal{L}$.

For the case $I = 0$, it follows from (55),(18) and (53) , we reach

$$\|Uf(x) - f(x)\| \leq \mathcal{L} \ L(x, 0), x \in \mathcal{B}_1. \tag{56}$$

Hence,

$$d(Uf, f) \leq \mathcal{L}^{1-0}, \qquad f \in \mathcal{A}. \tag{57}$$

For the case $I = 1$, it follows from (55),(23) and (53), we get

$$\|f\,(x) - Uf(x)\| \leq L(x, 0), \qquad x \in \mathcal{B}_1. \tag{58}$$

Thus, we obtain

$$d(f, Uf) \leq \mathcal{L}^{1-1}, \qquad f \in \mathcal{A}. \tag{59}$$

Hence, from (57) and (59), we arrive

$$d(Uf, f) \leq \mathcal{L}^{1-I}, \qquad f \in \mathcal{A}, \tag{60}$$

where $I = 0, 1$. Hence, property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{C}$ of U in $\mathcal{A}$ such that

$$\mathcal{C}(x) = \lim_{N \to \infty} \frac{1}{\ell_I^{3N}} f(\ell_I^{3N} x) \tag{61}$$

for all $x \in \mathcal{B}_1$. In order to show that $\mathcal{C}$ satisfies (1), replacing (x, y) by $(\ell_I^N x, \ell_I^N y)$ and dividing by ℓ_I^{3N} in (11), we have

$$\frac{1}{\ell_I^{3N}} \left\| F_k(\ell_I^N x, \ell_I^N y) \right\| \leq \frac{1}{2^{3N}} L(\ell_I^N x, \ell_I^N y)$$

for all $x, y \in \mathcal{B}_1$. Approaching $N \to \infty$ and using the definition of $\mathcal{C}$, (51) in the above inequality, we can see that $\mathcal{C}(x)$ satisfies the functional equation (1) for all $x \in \mathcal{B}_1$. By property (FP3), $\mathcal{C}$ is the unique fixed point of U in the set

$$\Delta = \{\mathcal{C} \in \mathcal{A} : d(f, \mathcal{C}) < \infty\},$$

such that

$$\|f(x) - \mathcal{C}(x)\| \leq \omega L(x, 0), x \in \mathcal{B}_1.$$

Finally by property (FP4), we obtain

$$\|f(x) - \mathcal{C}(x)\| \leq \|f(x) - Uf(x)\|, x \in \mathcal{B}_1.$$

This implies

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{\mathcal{L}^{1-I}}{1 - \mathcal{L}}$$

which yields

$$\|f(x) - \mathcal{C}(x)\| \leq \left(\frac{\mathcal{L}^{1-i}}{1 - \mathcal{L}}\right) L(x, 0), x \in \mathcal{B}_1.$$

So, the proof is completed. □

Using Theorem 3.7, we prove the following corollary concerning the stabilities of (1).

Corollary 3.8. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (27), for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{C}(x) - f(x)\| \leq \begin{cases} \dfrac{S\ 2^3}{\lambda|2^3 - 1|}; \\ \dfrac{S\ 2^3||x||^R}{\lambda|2^3 - 2^R|}; \\ \dfrac{S\ 2^3||x||^{2R}}{\lambda|2^3 - 2^{2R}|}; \end{cases} \tag{62}$$

for all $x \in \mathcal{B}_1$.

Proof. Let

$$L(x, y) = \begin{cases} S; \\ S\left(||x||^R + ||y||^R\right); \\ S\left(||x||^R||y||^R + ||x||^{2R} + ||y||^{2R}\right); \end{cases}$$

for all $x, y \in \mathcal{B}_1$. Now

$$\frac{1}{\ell_I^{3N}} L(\ell_I^N x, \ell_I^N y) = \begin{cases} \frac{S}{\ell_I^{3N}}, \\ \frac{S}{\ell_I^{3N}} \left\{ ||\ell_I^N x||^R + ||\ell_I^N y||^R \right\}, \\ \frac{S}{\ell_I^{3N}} \left\{ ||\ell_I^N x||^R \; ||\ell_I^N y||^R + \left\{ ||\ell_I^N x||^{2R} + ||\ell_I^N y||^{2R} \right\} \right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } N \to \infty, \\ \to 0 \text{ as } N \to \infty, \\ \to 0 \text{ as } N \to \infty. \end{cases}$$

Thus, (51) holds. But, we have

$$L(x, 0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right)$$

has the property

$$\frac{1}{\ell_I^3} L(\ell_I x, 0) = \mathcal{L} \; L(x, 0)$$

for all $x \in \mathcal{B}_1$. Hence,

$$L(x, 0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right) = \begin{cases} \frac{S \; 2^3}{\lambda}, \\ \frac{S \; 2^3}{\lambda \cdot 2^R} ||x||^R, \\ \frac{S \; 2^3}{\lambda \cdot 2^{2R}} ||x||^{2R} \end{cases} \tag{63}$$

for all $x \in \mathcal{B}_1$. It follows from (63),

$$\frac{1}{\ell_I^3} L(\ell_I x, 0) = \begin{cases} \ell_I^{-3} \; \frac{S \; 2^3}{\lambda}, \\ \ell_I^{R-3} \frac{S \; 2^3}{\lambda} ||x||^R \\ \ell_I^{2R-3} \frac{S \; 2^3}{\lambda} ||x||^{2R}. \end{cases}$$

Hence, the inequality (83) holds for

(i). $\mathcal{L} = \ell_I^{-3}$ if $I = 0$ and $\mathcal{L} = \frac{1}{\ell_I^{-3}}$ if $I = 1$;

(ii). $\mathcal{L} = \ell_I^{R-3}$ for $R < 3$ if $I = 0$ and $\mathcal{L} = \frac{1}{\ell_I^{R-3}}$ for $R > 3$ if $I = 1$;

(iii). $\mathcal{L} = \ell_I^{2R-3}$ for $2R > 3$ if $I = 0$ and $\mathcal{L} = \frac{1}{\ell_I^{2R-3}}$ for $2R > 3$ if $I = 1$.

Now, from (83), we prove the following cases for condition (i).

$$\begin{array}{ll}
\mathcal{L} = \ell_I^{-3}, I = 0 & \mathcal{L} = \frac{1}{\ell_I^{-3}}, I = 1 \\
\mathcal{L} = 2^{-3}, I = 0 & \mathcal{L} = \frac{1}{2^{-3}}, I = 1 \\
\mathcal{L} = 2^{-3}, I = 0 & \mathcal{L} = 2^{3}, I = 1 \\
\| f(x) - \mathcal{C}(x) \| & \| f(x) - \mathcal{C}(x) \| \\
\quad \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(x,0) & \quad \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(x,0) \\
\quad = \left(\frac{(2^{-3})^{1-0}}{1-2^{-3}}\right) \cdot \frac{S\ 2^3}{\lambda} & \quad = \left(\frac{(2^{3})^{1-1}}{1-2^{3}}\right) \cdot \frac{S\ 2^3}{\lambda} \\
\quad = \left(\frac{2^{-3}}{1-2^{-3}}\right) \cdot \frac{S\ 2^3}{\lambda} & \quad = \left(\frac{1}{1-2^{3}}\right) \cdot \frac{S\ 2^3}{\lambda} \\
\quad = \left(\frac{S\ 2^3}{\lambda(2^3-1)}\right) & \quad = \left(\frac{S\ 2^3}{\lambda(1-2^3)}\right)
\end{array}$$

Also, from (83), we prove the following cases for condition (ii).

$$\begin{array}{ll}
\mathcal{L} = \ell_I^{R-3}, R < 3, I = 0 & \mathcal{L} = \frac{1}{\ell_I^{R-3}}, R > 3, I = 1 \\
\mathcal{L} = 2^{R-3}, R < 3, I = 0 & \mathcal{L} = \frac{1}{2^{R-3}}, R < 3, I = 1 \\
\mathcal{L} = 2^{R-3}, R < 3, I = 0 & \mathcal{L} = 2^{3-R}, R > 3, I = 1 \\
\| f(x) - \mathcal{C}(x) \| & \| f(x) - \mathcal{C}(x) \| \\
\quad \leq \left(\frac{\mathcal{L}^{1-i}}{1-\mathcal{L}}\right) L(x,0) & \quad \leq \left(\frac{\mathcal{L}^{1-i}}{1-\mathcal{L}}\right) L(x,0) \\
\quad = \left(\frac{(2^{R-3})^{1-0}}{1-2^{R-3}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R & \quad = \left(\frac{(2^{3-R})^{1-1}}{1-2^{3-R}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R \\
\quad = \left(\frac{2^{R-3}}{1-2^{R-3}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R & \quad = \left(\frac{1}{1-2^{3-R}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R \\
\quad = \left(\frac{2^{R}}{2^3-2^{R}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R & \quad = \left(\frac{2^{R}}{2^R-2^{3}}\right) \cdot \frac{S\ 2^3}{\lambda\cdot 2^R}||x||^R
\end{array}$$

Finally, the proof of (83) for condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

Theorem 3.9. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (33) for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{N\to\infty} \frac{L(\ell_I^N x, \ell_I^N y)}{\ell_I^{4N}} = 0 \tag{64}$$

where

$$\ell_I = \begin{cases} k & if \quad I = 0, \\ \frac{1}{k} & if \quad I = 1 \end{cases} \tag{65}$$

holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function

$$L\left(x,0\right) = \frac{1}{k} L\left(\frac{x}{k}, 0\right)$$

with the property

$$\frac{1}{\ell_I^4} L(\ell_I x, 0) = \mathcal{L} L(w,0) \tag{66}$$

for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{Q}(x) - f(x)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(x, 0) \tag{67}$$

for all $x \in \mathcal{B}_1$.

Proof. The proof of the theorem is similar to that of Theorem 3.7 by define the linear mapping as $U : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Uf(x) = \frac{1}{\ell_I^4} f_q(\ell_I x),$$

for all $x \in \mathcal{B}_1$. For given $f, f_q \in \mathcal{A}$ □

Using Theorem 4.9, we prove the following corollary concerning the stabilities of (1).

Corollary 3.10. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (40), for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{Q}(x) - f(x)\| \leq \begin{cases} \dfrac{S}{2|k^4 - 1|}; \\ \dfrac{S|x||^R}{2|k^4 - k^R|}; \\ \dfrac{S||x||^{2R}}{2|k^4 - k^{2R}|}; \end{cases} \tag{68}$$

for all $x \in \mathcal{B}_1$.

Proof. The proof of the corollary is similar ideas of to that of Corollary 3.8. Hence the details of the proof are omitted. □

Theorem 3.11. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality (44), for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satisfies the conditions (51) and (64), where ℓ_I is respectively defined in (52) and (65) that holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the functions*

$$L(x, 0) = \frac{1}{2} L\left(\frac{x}{2}, 0\right) \quad \textit{and} \quad L(x, 0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right)$$

satisfies the properties (53), (66) and

$$\frac{1}{\ell_I^3} L(\ell_I x, 0) = \mathcal{L} L(w, 0) \tag{69}$$

for all $x \in \mathcal{B}_1$. Then there are only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right)(L(x,0) + L(-x,0)) \tag{70}$$

for all $x \in \mathcal{B}_1$.

Proof. By the definition of $f_c(x)$ and $f_q(x)$ in Theorem 3.5 and with the help of Theorems 3.7 and 4.9, we arrive our desired result. □

Using Theorem 3.11, we prove the following corollary concerning the stabilities of (1).

Corollary 3.12. *Assume that S and R be positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (49) for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q}(x) : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \begin{cases} \dfrac{S\ 2^3}{\lambda|2^3-1|} + \dfrac{S}{2|k^4-1|}; \\ \dfrac{S\ 2^3||x||^R}{\lambda|2^3-2^R|} + \dfrac{S|x||^R}{2|k^4-k^R|}; \\ \dfrac{S\ 2^3||x||^{2R}}{\lambda|2^3-2^{2R}|} + \dfrac{S||x||^{2R}}{2|k^4-k^{2R}|}; \end{cases} \tag{71}$$

for all $x \in \mathcal{B}_1$.

4. Stability in quasi beta Banach space

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1) in Quasi Beta Banach Space. To prove stability results, let us take $\mathcal{B}_1$ be a normed space and $\mathcal{B}_2$ be a Quasi Beta Banach Space.

4.1. Hyers Method.

Theorem 4.1. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (11) for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0,\infty)$ with the condition (12) for all $x \in \mathcal{B}_1$. Then there exists only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{C}(x) - f(x)\| \leq \frac{K^{N-1}}{\lambda^\beta} \sum_{M=\frac{1-J}{2}}^{\infty} \frac{L(2^{MJ}x, 0)}{2^{3MJ}} \tag{72}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$ and $\lambda = 2k^2(k^2-1)$, where the mapping $\mathcal{C}$ is defined in (14) for all $x \in \mathcal{B}_1$.

Proof. From (16), we arrive

$$\left\|-\frac{k^2}{4}(k^2-1)f(2x)+2k^2\left(k^2-1\right)f(x)\right\|\leq L(x,0) \tag{73}$$

for all $x\in\mathcal{B}_1$. It follows from (73) and since $k\neq 0,\pm 1$ that

$$\left\|\frac{f(2x)}{2^3}-f(x)\right\|\leq\frac{L(x,0)}{\left(2k^2\left(k^2-1\right)\right)^{\beta}} \tag{74}$$

for all $x\in\mathcal{B}_1$. Let $\lambda=2k^2\left(k^2-1\right)$ in above inequality, we arrive

$$\left\|\frac{f(2x)}{2^3}-f(x)\right\|\leq\frac{L(x,0)}{\lambda^{\beta}} \tag{75}$$

for all $x\in\mathcal{B}_1$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.1 concerning the stabilities of (1).

Corollary 4.2. *Assume that S and R are positive numbers. Let $f:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be an odd function fulfilling the inequality (27) for all $x,y\in\mathcal{B}_1$. Then there exists a one and only cubic mapping $\mathcal{C}:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{C}(x)-f(x)\|\leq\begin{cases}\dfrac{K^{N-1}S}{\lambda^{\beta}|2^3-1|};\\ \dfrac{K^{N-1}S||x||^{R}}{\lambda^{\beta}|2^3-2^{R\beta}|};\\ \dfrac{K^{N-1}S||x||^{2R}}{\lambda^{\beta}|2^3-2^{2R\beta}|};\end{cases} \tag{76}$$

for all $x\in\mathcal{B}_1$.

Theorem 4.3. *Let $f:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be an even function fulfilling the inequality (33) for all $x,y\in\mathcal{B}_1$, where $L:\mathcal{B}_1^2\longrightarrow[0,\infty)$ satifies the condition (34) for all $x\in\mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q}:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{Q}(x)-f(x)\|\leq\frac{K^{N-1}}{(2\ k^4)^{\beta}}\sum_{M=\frac{1-J}{2}}^{\infty}\frac{L(k^{MJ}x,0)}{k^{4MJ}} \tag{77}$$

for all $x\in\mathcal{B}_1$ with $J=\pm 1$, where the mapping $\mathcal{Q}(x)$ is defined in (36) for all $x\in\mathcal{B}_1$.

Proof. From (38), we arrive

$$\left\|2f(kx)-2k^4f(x)\right\|\leq L(x,0) \tag{78}$$

for all $x \in \mathcal{B}_1$. It follows from (78) and since $k \neq 0, \pm 1$ that

$$\left\| \frac{f(kx)}{k^4} - f(x) \right\| \leq \frac{L(x,0)}{(2\ k^4)^\beta} \tag{79}$$

for all $x \in \mathcal{B}_1$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.3 concerning the stabilities of (1).

Corollary 4.4. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (40) for all $x, y \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q}(x) : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{Q}(x) - f(x)\| \leq \begin{cases} \dfrac{K^{N_1} k^4\ S}{(2k^4)^\beta\, |k^4 - 1|}; \\ \dfrac{K^{N_1} k^4\ S||x||^R}{(2k^4)^\beta\, |k^4 - k^{R\beta}|}; \\ \dfrac{K^{N_1} k^4\ S||x||^{2R}}{(2k^4)^\beta\, |k^4 - k^{2R\beta}|}; \end{cases} \tag{80}$$

for all $x \in \mathcal{B}_1$.

Theorem 4.5. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality (44) for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satifies the conditions (12) and (34) for all $x \in \mathcal{B}_1$. Then there are only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\begin{aligned} \|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \frac{K^N}{2^\beta} \Bigg\{ &\frac{1}{\lambda^\beta} \sum_{M=\frac{1-J}{2}}^{\infty} \left(\frac{L(2^{MJ}x, 0)}{2^{3MJ}} + \frac{L(-2^{MJ}x, 0)}{2^{3MJ}} \right) \\ &+ \frac{1}{(2k^4)^\beta} \sum_{M=\frac{1-J}{2}}^{\infty} \left(\frac{L(k^{MJ}x, 0)}{k^{4MJ}} + \frac{L(-k^{MJ}x, 0)}{k^{4MJ}} \right) \Bigg\} \end{aligned} \tag{81}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$, where the mappings $\mathcal{C}(x)$ and $\mathcal{Q}(x)$ are defined in (14) and (36), respectively, for all $x \in \mathcal{B}_1$.

Proof. The proof is similar lines to that of Theorem 3.5. □

The following corollary is an immediate consequence of Theorem 4.5 concerning the stabilities of (1).

Corollary 4.6. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality (49) for all $x, y \in \mathcal{B}_1$. Then there are only one* *cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \begin{cases} \dfrac{K^N \; S}{2^\beta}\left(\dfrac{1}{\lambda^\beta|2^3-1|} + \dfrac{k^4}{(2k^4)^\beta\,|k^4-1|}\right); \\ \dfrac{K^N \; S||x||^R}{2^\beta}\left(\dfrac{1}{\lambda|2^3-2^{R\beta}|} + \dfrac{k^4}{(2k^4)^\beta\,|k^4-k^{R\beta}|}\right); \\ \dfrac{K^N \; S||x||^{2R}}{2^\beta}\left(\dfrac{1}{\lambda|2^3-2^{2R\beta}|} + \dfrac{k^4}{(2k^4)^\beta\,|k^4-k^{2R\beta}|}\right); \end{cases} \tag{82}$$

for all $x \in \mathcal{B}_1$.

4.2. Radu's method. The proof of following theorems and corollaries are similar lines to that of Section 3.2.

Theorem 4.7. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (11), for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satisfies the condition (51) and ℓ_I is defined in (52) such that holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function*

$$L(x, 0) = \frac{2^3}{\lambda^\beta} L\left(\frac{x}{2}, 0\right)$$

satisfies the property (53), for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{C}(x) - f(x)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(x, 0) \tag{83}$$

for all $x \in \mathcal{B}_1$.

Corollary 4.8. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (27), for all $x \in \mathcal{B}_1$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{C}(x) - f(x)\| \leq \begin{cases} \dfrac{S\;2^3}{\lambda^\beta|2^3-1|}; \\ \dfrac{S\;2^3||x||^R}{\lambda^\beta|2^3-2^{R\beta}|}; \\ \dfrac{S\;2^3||x||^{2R}}{\lambda^\beta|2^3-2^{2R\beta}|}; \end{cases} \tag{84}$$

for all $x \in \mathcal{B}_1$.

Theorem 4.9. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (33), for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satisfies the condition (64) and ℓ_I*

is defined in 4FMaq.1a such that holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function

$$L(x,0) = \frac{1}{2^\beta} L\left(\frac{x}{2}, 0\right)$$

with the property (66) for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|\mathcal{Q}(x) - f(x)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) L(w,0) \tag{85}$$

for all $x \in \mathcal{B}_1$.

Corollary 4.10. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (40), for all $x \in \mathcal{B}_1$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|\mathcal{Q}(x) - f(x)\| \leq \begin{cases} \dfrac{S}{(2k^4)^\beta |k^4 - 1|}; \\ \dfrac{S|x||^R}{(2k^4)^\beta |k^4 - k^{R\beta}|}; \\ \dfrac{S||x||^{2R}}{(2k^4)^\beta |k^4 - k^{2R\beta}|}; \end{cases} \tag{86}$$

for all $x \in \mathcal{B}_1$.

Theorem 4.11. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality (44), for all $x, y \in \mathcal{B}_1$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ satifies the conditions (51) and (64), and ℓ_I is respectively defined in (52) and (65) that holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the functions*

$$L(x,0) = \frac{1}{2^\beta} L\left(\frac{x}{2}, 0\right) \quad \textit{and} \quad L(x,0) = \frac{2^3}{\lambda^\beta} L\left(\frac{x}{2}, 0\right)$$

with the properties (53) and (66), for all $x \in \mathcal{B}_1$. Then there are only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \frac{K}{2^\beta} \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) (L(x,0) + L(-x,0)) \tag{87}$$

for all $x \in \mathcal{B}_1$.

Corollary 4.12. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality (49), for all $x \in \mathcal{B}_1$. Then there is only* *one quartic mapping $\mathcal{Q}(x) : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and*

$$\|f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\| \leq \begin{cases} \frac{K}{2^\beta}\left(\frac{S\ 2^3}{\lambda^\beta|2^3-1|} + \frac{S}{(2k^4)^\beta\,|k^4-1|}\right); \\ \frac{K}{2^\beta}\left(\frac{S\ 2^3||x||^R}{\lambda^\beta|2^3-2^{R\beta}|} + \frac{S|x||^R}{(2k^4)^\beta\,|k^4-k^{R\beta}|}\right); \\ \frac{K}{2^\beta}\left(\frac{S\ 2^3||x||^{2R}}{\lambda^\beta|2^3-2^{2R\beta}|} + \frac{S||x||^{2R}}{(2k^4)^\beta\,|k^4-k^{2R\beta}|}\right); \end{cases} \tag{88}$$

for all $x \in \mathcal{B}_1$.

5. Stability in fuzzy Banach space

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1) in fuzzy Banach spaces. To prove stability results, let us take $\mathcal{B}_3, (\mathcal{B}_1, N)$ and $(\mathcal{B}_2, N')$ are linear space, fuzzy normed space and fuzzy Banach space, respectively.

5.1. Definitions on fuzzy Banach spaces.

Definition 5.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ (so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,
$(FNS1)$ $N(x, c) = 0$, for $c \leq 0$;
$(FNS2)$ $x = 0$ if and only if $N(x, c) = 1$, for all $c > 0$;
$(FNS3)$ $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
$(FNS4)$ $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
$(FNS5)$ $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
$(FNS6)$ for $x \neq 0, N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 5.2. Let $(X, || \cdot ||)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 5.3. Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.

Definition 5.4. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists n_0 such that, for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 5.5. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and this fuzzy normed space is called a fuzzy Banach space.

Definition 5.6. A mapping $f : X \longrightarrow Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if, for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$, then f is said to be continuous on X.

5.2. Hyers method.

Theorem 5.1. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd mapping fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y), s\right) \geq \mathcal{N}'\left(L\left(x,y\right), s\right) \tag{89}$$

for all $x, y \in \mathcal{B}_1$ and $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow \mathcal{B}_3$, $0 < \left(\dfrac{s}{2^3}\right)^J < 1$ and the following conditions hold

$$\lim_{N\to\infty} \mathcal{N}'\left(L\left(2^{JN}x, 2^{JN}y\right), 2^{3JN}s\right) = 1 \tag{90}$$

$$\mathcal{N}'\left(L\left(2^J z, 2^J y\right), s\right) \geq \mathcal{N}'\left(t^J L\left(x,y\right), s\right) \tag{91}$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Then there is a unique cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ which satisfies (1) and

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}'\left(L\left(x,0\right), \frac{s|2^3 - t|}{2^3\ \eta}\right) \tag{92}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$ and $\lambda = 2k^2\left(k^2 - 1\right)$. The mapping $\mathcal{C}$ is defined by

$$\lim_{N\to\infty} \mathcal{N}\left(\mathcal{C}(x) - \frac{f(2^s x)}{2^{3N}}, s\right) = 1 \tag{93}$$

for all $x \in \mathcal{B}_1$.

Proof. Changing y by 0 in (89), we reach

$$\begin{aligned} \mathcal{N}\Big(f(x) + f(kx) + f(x) + f(-kx) - k^2\left[2f(x) + f(x) + f(-x)\right] \\ +2(k^4 - 1)f(x) - \frac{k^2}{4}(k^2 - 1)f(2x), s\Big) \geq N'\left(L\left(x,0\right), s\right) \end{aligned} \tag{94}$$

for all $x \in \mathcal{B}_1$. Using oddness of f in (94) and it follows from (94) that

$$\mathcal{N}\left(2f(x) - 2k^2 f(x) + 2(k^4-1)f(x) - \frac{k^2}{4}(k^2-1)f(2x), s\right) \geq N'\left(L\left(x,0\right),s\right);$$

or

$$\mathcal{N}\left(-\frac{k^2}{4}(k^2-1)f(2x) + 2k^2\left(k^2-1\right)f(x), s\right) \geq N'\left(L\left(x,0\right),s\right) \tag{95}$$

for all $x \in \mathcal{B}_1$. It follows from (95) and $k \neq 0, \pm 1$,

$$\mathcal{N}\left(\frac{f(2x)}{2^3} - f(x), \frac{s}{2k^2\left(k^2-1\right)}\right) \geq \mathcal{N}'\left(L\left(x,0\right),s\right) \tag{96}$$

for all $x \in \mathcal{B}_1$. Set $\lambda = 2k^2\left(k^2-1\right)$ in the above inequality, we have

$$\mathcal{N}\left(\frac{f(2x)}{2^3} - f(x), \frac{s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(x,0\right),s\right) \tag{97}$$

for all $x \in \mathcal{B}_1$. Replacing x by $2^N x$ in (97), we obtain

$$\mathcal{N}\left(\frac{f(2^{N+1}x)}{2^3} - f(2^N x), \frac{s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(2^N x,0\right),s\right) \tag{98}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Using (91), $(FNS3)$ in (98), we arrive

$$\mathcal{N}\left(\frac{f(2^{N+1}x)}{2^3} - f(2^N x), \frac{s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(x,0\right),\frac{s}{t^N}\right) \tag{99}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. It is easy to verify from (99), that

$$\mathcal{N}\left(\frac{f(2^{N+1}x)}{2^{3(N+1)}} - \frac{f(2^N x)}{2^{3N}}, \frac{s}{\lambda\ 2^{3N}}\right) \geq \mathcal{N}'\left(L\left(x,0\right),\frac{s}{t^N}\right) \tag{100}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Switching s by $t^N s$ in (100), we get

$$\mathcal{N}\left(\frac{f(2^{N+1}x)}{2^{3(q+1)}} - \frac{f(2^N x)}{2^{3N}}, \left(\frac{t}{2^3}\right)^N \frac{s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(x,0\right),s\right) \tag{101}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. It is easy to see that

$$\frac{f(2^N x)}{2^{3N}} - f(x) = \sum_{r=0}^{N-1}\left[\frac{f(2^{r+1}x)}{2^{3(r+1)}} - \frac{f(2^r x)}{2^{3r}}\right] \tag{102}$$

for all $x \in \mathcal{B}_1$. From equations (101) and (102), we have

$$\begin{aligned}\mathcal{N}\left(\frac{f(2^N x)}{2^{3N}} - f(x), \sum_{r=0}^{N-1}\left(\frac{t}{2^3}\right)^r \frac{s}{\lambda}\right) &\geq \min \bigcup_{r=0}^{N-1}\left\{\mathcal{N}\left(\frac{f(2^{r+1}x)}{2^{3(r+1)}} - \frac{f(2^r x)}{2^{3r}}, \left(\frac{t}{2^3}\right)^r \frac{s}{\lambda}\right)\right\} \\ &\geq \min \bigcup_{r=0}^{N-1}\left\{N'\left(L\left(x,0\right),s\right)\right\} = \mathcal{N}'\left(L\left(x,0\right),s\right)\end{aligned} \tag{103}$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. Replacing x by $2^P x$ in (103) and using (91), $(FNS3)$, and substituting s by $t^P s$, we obtain

$$\mathcal{N}\left(\frac{f(2^{N+P}x)}{2^{3(N+P)}} - \frac{f(2^P x)}{2^{3P}}, \sum_{r=P}^{N+P-1} \left(\frac{t}{2^3}\right)^r \frac{s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(x,0\right),s\right) \tag{104}$$

for all $x \in \mathcal{B}_1$ and all $s > 0$ and all $P > N \geq 0$. Using $(FNS3)$ in (104), we obtain

$$\mathcal{N}\left(\frac{f(2^{N+P}x)}{2^{3(N+P)}} - \frac{f(2^P x)}{2^{3P}}, s\right) \geq \mathcal{N}'\left(L\left(x,0\right), \frac{s\ \lambda}{\sum\limits_{r=P}^{N+P-1} \left(\frac{t}{2^3}\right)^r}\right) \tag{105}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Since $0 < t < 2^3$ and $\sum\limits_{r=0}^{N} \left(\frac{t}{2^3}\right)^r < \infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\frac{f(2^N x)}{2^{3N}}\right\}$ is a Cauchy sequence in $(\mathcal{B}_2, N')$. Since $(\mathcal{B}_2, N')$ is a fuzzy Banach space, this sequence converges to some point $\mathcal{C}(x) \in \mathcal{B}_2$. So one can define the mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ by

$$\lim_{N\to\infty} \mathcal{N}\left(\mathcal{C}(x) - \frac{f(2^N x)}{2^{3N}}, s\right) = 1 \tag{106}$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. Letting $P = 0$ and $N \to \infty$ in (105), we get

$$\mathcal{N}\left(\mathcal{C}(x) - f(x), s\right) \geq \mathcal{N}'\left(L\left(x,0\right), \frac{s\ \lambda(2^3 - t)}{2^3}\right)$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. To prove $\mathcal{C}$ satisfies the (1), replacing (x, y) by $(2^N x, 2^N y)$ in (89), we obtain

$$\mathcal{N}\left(\mathcal{C}(x,y), s\right) = \mathcal{N}\left(\frac{1}{2^{3N}} F_k(2^N x, 2^N y), s\right) \geq \mathcal{N}'\left(L\left(2^N x, 2^N y\right), 2^{3N} s\right) \tag{107}$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Now,

$$
\begin{aligned}
N\Big(&\mathcal{C}(x+ky)+\mathcal{C}(kx+y)+\mathcal{C}(x-ky)+\mathcal{C}(y-kx)\\
&\qquad -k^2\left[2\mathcal{C}(x+y)-\mathcal{C}(x-y)-\mathcal{C}(y-x)\right]\\
&\qquad -2(k^4-1)[\mathcal{C}(x)+\mathcal{C}(y)]-\frac{k^2}{4}(k^2-1)[\mathcal{C}(2x)+\mathcal{C}(2y)], s\Big)\\
\geq \min\Bigg\{&\mathcal{N}\left(\mathcal{C}(x+ky)-\frac{1}{2^{3N}}f(2^N(x+ky)), \frac{s}{12}\right),\\
&\mathcal{N}\left(\mathcal{C}(kx+y)-\frac{1}{2^{3N}}f(2^N(kx+y)), \frac{s}{12}\right),\\
&\mathcal{N}\left(\mathcal{C}(x-ky)-\frac{1}{2^{3N}}f(2^N(x-ky)), \frac{s}{12}\right),\\
&\mathcal{N}\left(\mathcal{C}(y-kx)-\frac{1}{2^{3N}}f(2^N(y-kx)), \frac{s}{12}\right),\\
&\mathcal{N}\left(k^2\left[2\mathcal{C}(x+y)-\frac{2}{2^{3N}}f(2^N(x+y))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(k^2\left(-\mathcal{C}(x-y)+\frac{1}{2^{3N}}f(2^N(x-y))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(k^2\left[-\mathcal{C}(y-x)+\frac{1}{2^{3N}}f(2^N(y-x))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(-2(k^4-1)\left[\mathcal{C}(x)+\frac{1}{2^{3N}}f(2^N(x))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(-2(k^4-1)\left[\mathcal{C}(y)+\frac{1}{2^{3N}}f(2^N(y))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(-\frac{k^2}{4}(k^2-1)\left[\mathcal{C}(2x)+\frac{1}{2^{3N}}f(2^N(2x))\right], \frac{s}{12}\right),\\
&\mathcal{N}\left(-\frac{k^2}{4}(k^2-1)\left[\mathcal{C}(2y)+\frac{1}{2^{3N}}f(2^N(2y))\right], \frac{s}{12}\right),\\
&\mathcal{N}\Bigg(\frac{1}{2^{3N}}f(2^N(x+ky))+\frac{1}{2^{3N}}f(2^N(kx+y))+\frac{1}{2^{3N}}f(2^N(x-ky))\\
&+\frac{1}{2^{3N}}f(2^N(y-kx))k^2\left[\frac{2}{2^{3N}}f(2^N(x+y))-\frac{1}{2^{3N}}f(2^N(x-y))\right.\\
&\left.-\frac{2}{2^{3N}}f(2^N(y-x))\right]-2(k^4-1)\left[\frac{1}{2^{3N}}f(2^N(x))+\frac{1}{2^{3N}}f(2^N(y))\right]\\
&-\frac{k^2}{4}(k^2-1)\left[\frac{1}{2^{3N}}f(2^N(2x))+\frac{1}{2^{3N}}f(2^N(2y))\right], \frac{s}{12}\Bigg)\Bigg\}
\end{aligned}
\tag{108}
$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Using (106), (107), $(FNS5)$ and (108), we reach

$$N\Big(\mathcal{C}(x+ky)+\mathcal{C}(kx+y)+\mathcal{C}(x-ky)+\mathcal{C}(y-kx) \\ - k^2\left[2\mathcal{C}(x+y)-\mathcal{C}(x-y)-\mathcal{C}(y-x)\right] \\ - 2(k^4-1)[\mathcal{C}(x)+\mathcal{C}(y)] - \frac{k^2}{4}(k^2-1)[\mathcal{C}(2x)+\mathcal{C}(2y)], s\Big) \\ \geq \min\left\{1,1,1,1,1,1,1,1,1,1,1,1,\mathcal{N}'\left(L\left(2^N x, 2^N y\right), 2^{3N}s\right)\right\} \quad (109)$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Approaching N tends to infinity in (111) and applying (91), we get

$$N\Big(\mathcal{C}(x+ky)+\mathcal{C}(kx+y)+\mathcal{C}(x-ky)+\mathcal{C}(y-kx)-k^2[2\mathcal{C}(x+y)-\mathcal{C}(x-y) \\ -\mathcal{C}(y-x)]-2(k^4-1)[\mathcal{C}(x)+\mathcal{C}(y)]-\frac{k^2}{4}(k^2-1)[\mathcal{C}(2x)+\mathcal{C}(2y)], s\Big) = 1 \quad (110)$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Using $(FNS2)$ in (110), it gives

$$\mathcal{C}(x+ky)+\mathcal{C}(kx+y)+\mathcal{C}(x-ky)+\mathcal{C}(y-kx) \\ = k^2\left[2\mathcal{C}(x+y)+\mathcal{C}(x-y)+\mathcal{C}(y-x)\right] \\ + 2(k^4-1)[\mathcal{C}(x)+\mathcal{C}(y)] + \frac{k^2}{4}(k^2-1)[\mathcal{C}(2x)+\mathcal{C}(2y)]$$

for all $x, y \in \mathcal{B}_1$. Hence $\mathcal{C}$ satisfies the functional equation (1). The existence of $\mathcal{C}$ is unique. Indeed, if $\mathcal{C}'$ be another cubic functional equation satisfying (1) and (93). So,

$$N(\mathcal{C}(x)-\mathcal{C}'(x), s) = \mathcal{N}\left(\frac{\mathcal{C}(2^N x)}{2^{3N}} - \frac{\mathcal{C}'(2^N x)}{2^{3N}}, s\right) \\ \geq \min\left\{\mathcal{N}\left(\frac{\mathcal{C}(2^N x)}{2^{3N}} - \frac{f(2^N x)}{2^{3N}}, \frac{s}{2}\right), \mathcal{N}\left(\frac{\mathcal{C}'(2^N x)}{2^{3N}} - \frac{f(2^N x)}{2^{3N}}, \frac{s}{2}\right)\right\} \\ \geq \mathcal{N}'\left(L\left(2^N z, 2^N 0\right), \frac{s\ \lambda(2^3-t)2^{3N}}{2^3\ 2}\right) \\ = \mathcal{N}'\left(L\left(x, 0\right), \frac{s\ \lambda(2^3-t)2^{3N}}{2^3 t^N\ 2}\right)$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. Since

$$\lim_{N\to\infty} \frac{s\ \lambda(2^3-t)2^{3N}}{2^3 t^N\ 2} = \infty,$$

we obtain

$$\lim_{N\to\infty} \mathcal{N}'\left(L\left(x, 0\right), \frac{s\ \lambda(2^3-t)2^{3N}}{2^3 t^N\ 2}\right) = 1.$$

Thus

$$N(\mathcal{C}(x)-\mathcal{C}'(x), s) = 1$$

for all $x \in \mathcal{B}_1$ and all $s > 0$, hence $\mathcal{C}(x) = \mathcal{C}'(x)$. Therefore $\mathcal{C}(x) - \mathcal{C}'(x)$ is unique. Hence for $J = 1$ the theorem holds.

Replacing x by $\frac{x}{2}$ in (97), we achieve

$$\mathcal{N}\left(f(x) - 2^3\left(\frac{x}{2}\right), \frac{2^3\ s}{\lambda}\right) \geq \mathcal{N}'\left(L\left(\frac{x}{2}, 0\right), s\right) \tag{111}$$

for all $x \in \mathcal{B}_1$. The rest of the proof is similar ideas to that of case $J = 1$. Thus the theorem is true for $J = -1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 5.1 concerning the stabilities of (1).

Corollary 5.2. *Assume that S and R be positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y), s\right) \geq \begin{cases} \mathcal{N}'\left(S, s\right) \\ \mathcal{N}'\left(S\left\{||x||^R + ||y||^R\right\}, s\right), & R \neq 3; \\ \mathcal{N}'\left(S\left\{||x||^R||y||^R + \left\{||x||^{2R} + ||y||^{2R}\right\}\right\}, s\right), & 2R \neq 3; \end{cases} \tag{112}$$

for all $x, y \in \mathcal{B}_1$, then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \begin{cases} \mathcal{N}'\left(2^3\ S, s\ \lambda|2^3 - 1|\right), \\ \mathcal{N}'\left(2^3\ S||x||^R, s\ \lambda|2^3 - 2^R|\right), \\ \mathcal{N}'\left(2^3\ S||x||^{2R}, s\ \lambda|2^3 - 2^{2R}|\right) \end{cases} \tag{113}$$

for all $x \in \mathcal{B}_1$.

Theorem 5.3. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even mapping fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y), s\right) \geq \mathcal{N}'\left(L\left(x,y\right), s\right) \tag{114}$$

for all $x, y \in \mathcal{B}_1$ and $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow \mathcal{B}_3$ with $0 < \left(\frac{s}{k^4}\right)^J < 1$ and the conditions

$$\lim_{N\to\infty} \mathcal{N}'\left(L\left(k^{JN}x, k^{JN}y\right), k^{3JN}s\right) = 1 \tag{115}$$

$$\mathcal{N}'\left(L\left(k^J z, k^J y\right), s\right) \geq \mathcal{N}'\left(t^J L\left(x,y\right), s\right) \tag{116}$$

for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Then there is a unique quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ which satisfies (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \mathcal{N}'\left(L\left(x, 0\right), \frac{2s|k^4 - t|}{k^4}\right) \tag{117}$$

for all $x \in \mathcal{B}_1$ with $J = \pm 1$. The mapping $\mathcal{Q}(x)$ is defined by

$$\lim_{N\to\infty} \mathcal{N}\left(\mathcal{Q}(x) - \frac{f(k^s x)}{k^{4N}}, s\right) = 1 \tag{118}$$

for all $x \in \mathcal{B}_1$.

Proof. Changing y by 0 in (114), we reach

$$\mathcal{N}\Big(f(x)+f(kx)+f(x)+f(-kx)-k^2\left[2f(x)+f(x)+f(-x)\right] \\ +2(k^4-1)f(x)-\frac{k^2}{4}(k^2-1)f(2x),s\Big)\geq N'\left(L\left(x,0\right),s\right) \tag{119}$$

for all $x\in\mathcal{B}_1$. Using evenness of f in (119) and it follows from (119) that

$$\mathcal{N}\left(2f(kx)+2f(x)-4k^2f(x)+2(k^4-1)f(x)-\frac{k^2}{4}(k^2-1)f(2x),s\right) \\ \geq N'\left(L\left(x,0\right),s\right);$$

or

$$\mathcal{N}\left(2f(kx)+2f(x)-4k^2f(x)+2(k^4-1)f(x)-4\ k^2(k^2-1)f(x),s\right) \\ \geq N'\left(L\left(x,0\right),s\right);$$

or

$$\mathcal{N}\left(2f(kx)-2k^4f(x),s\right)\geq N'\left(L\left(x,0\right),s\right) \tag{120}$$

for all $x\in\mathcal{B}_1$. It follows from (120) and since $k\neq 0,\pm 1$ that

$$\mathcal{N}\left(\frac{f(2x)}{k^4}-f(x),\frac{s}{2k^4}\right)\geq \mathcal{N}'\left(L\left(x,0\right),s\right) \tag{121}$$

for all $x\in\mathcal{B}_1$. The rest of the proof is similar to that of Theorem 5.1. □

The following corollary is an immediate consequence of Theorem 5.3 concerning the stabilities of (1).

Corollary 5.4. *Assume that S and R are positive numbers. Let $f:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be an even function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y),s\right)\geq\begin{cases}\mathcal{N}'\left(S,s\right) & \\ \mathcal{N}'\left(S\left\{||x||^R+||y||^R\right\},s\right), & R\neq 4;\\ \mathcal{N}'\left(S\left\{||x||^R||y||^R+\left\{||x||^{2R}+||y||^{2R}\right\}\right\},s\right), & 2R\neq 4;\end{cases} \tag{122}$$

for all $x,y\in\mathcal{B}_1$, then there is only one quartic mapping $\mathcal{Q}:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x)-\mathcal{Q}(x),s\right)\geq\begin{cases}\mathcal{N}'\left(k^4\ S,2\ s|k^4-1|\right),\\ \mathcal{N}'\left(k^4\ S||x||^R,2\ s|k^4-k^R|\right),\\ \mathcal{N}'\left(k^4\ S||x||^{2R},2\ s|k^4-k^{2R}|\right)\end{cases} \tag{123}$$

for all $x\in\mathcal{B}_1$.

Theorem 5.5. *Let $f:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be a mapping fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y),s\right)\geq\mathcal{N}'\left(L\left(x,y\right),s\right) \tag{124}$$

for all $x, y \in \mathcal{B}_1$ and $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow \mathcal{B}_3$, $0 < \left(\frac{s}{2^3}\right)^J < 1$; $0 < \left(\frac{s}{k^4}\right)^J < 1$, and the conditions (90), (115), (91), (116) hold, for all $x, y \in \mathcal{B}_1$ and all $s > 0$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and a unique quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ which satisfies (1) and

$$\begin{aligned}
&\mathcal{N}\left(f(x) - \mathcal{C}(x) - \mathcal{Q}(x), s\right)\\
&\geq \min\left\{\mathcal{N}'\left(L(x,0), \frac{s|2^3 - t|}{2^3\ \eta}\right), \mathcal{N}'\left(L(-x,0), \frac{s|2^3 - t|}{2^3\ \eta}\right),\right.\\
&\qquad \left.\mathcal{N}'\left(L(x,0), \frac{2\ s|k^4 - t|}{k^4}\right), \mathcal{N}'\left(L(-x,0), \frac{2\ s|k^4 - t|}{k^4}\right)\right\} \qquad (125)
\end{aligned}$$

for all $x \in \mathcal{B}_1$ and $J = \pm 1$. The mappings $\mathcal{C}$ and $\mathcal{Q}$ are defined in (93) and (118), respectively.

Proof. If we define

$$f_C(x) = \frac{f(x) - f(-x)}{2} \qquad \text{for all} \qquad x \in \mathcal{B}_1$$

it follows that

$$f_C(0) = 0 \qquad \text{and} \qquad f_C(-x) = -f_C(x) \qquad \text{for all} \qquad x \in \mathcal{B}_1.$$

So, by definition of $f_C(x)$ it is easy to verify that

$$\mathcal{N}\left(F_{C_k}(x,y), s\right) \geq \min\left\{\mathcal{N}\left(f(x,y), s\right), \mathcal{N}\left(f(-x,-y), s\right)\right\} \qquad (126)$$

for all $x, y \in \mathcal{B}_1$ and $s > 0$. Hence, by Theorem 5.1,

$$\mathcal{N}\left(f_C(x) - \mathcal{C}(x), s\right) \geq \min\left\{\mathcal{N}'\left(L(x,0), \frac{s|2^3 - t|}{2^3\ \eta}\right), \mathcal{N}'\left(L(-x,0), \frac{s|2^3 - t|}{2^3\ \eta}\right)\right\} \qquad (127)$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Also, if we define

$$f_Q(x) = \frac{f(x) - f(-x)}{2} \qquad \text{for all } x \in \mathcal{B}_1$$

and it follows that

$$f_Q(0) = 0 \qquad \text{and} \qquad f_Q(-x) = f_Q(x) \qquad \text{for all} \qquad x \in \mathcal{B}_1.$$

So, by definition of $f_Q(x)$ it is easy to verify that

$$\mathcal{N}\left(F_{Q_k}(x,y), s\right) \geq \min\left\{\mathcal{N}\left(f(x,y), s\right), \mathcal{N}\left(f(-x,-y), s\right)\right\} \qquad (128)$$

for all $x, y \in \mathcal{B}_1$ and $s > 0$. Hence, by Theorem 5.3,

$$\mathcal{N}\left(f_Q(x) - \mathcal{Q}(x), s\right) \geq \min\left\{\mathcal{N}'\left(L(x,0), \frac{2\ s|k^4 - t|}{k^4}\right), \mathcal{N}'\left(L(-x,0), \frac{2\ s|k^4 - t|}{k^4}\right)\right\} \qquad (129)$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Define

$$f(x) = f_C(x) + f_Q(x) \tag{130}$$

for all $x \in \mathcal{B}_1$. Using (127), (129) in (130), we arrive

$$\begin{aligned}
&\mathcal{N}\left(f(x) - \mathcal{C}(x) - \mathcal{Q}(x), 2s\right) \\
&\quad = \mathcal{N}\left(f_C(x) + f_Q(x) - \mathcal{C}(x) - \mathcal{Q}(x), 2s\right) \\
&\quad \geq \min\left\{\mathcal{N}\left(f_C(x) - \mathcal{C}(x), s\right), \mathcal{N}\left(f_Q(x) - \mathcal{Q}(x), s\right)\right\} \\
&\quad \geq \min\left\{\mathcal{N}'\left(L\left(x, 0\right), \frac{s|2^3 - t|}{2^3\ \eta}\right), \mathcal{N}'\left(L\left(-x, 0\right), \frac{s|2^3 - t|}{2^3\ \eta}\right),\right. \\
&\qquad\qquad \left.\mathcal{N}'\left(L\left(x, 0\right), \frac{2\ s|k^4 - t|}{k^4}\right), \mathcal{N}'\left(L\left(-x, 0\right), \frac{2\ s|k^4 - t|}{k^4}\right)\right\}
\end{aligned}$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. $\square$

The following corollary is an immediate consequence of Theorem 5.5 concerning the stabilities of (1).

Corollary 5.6. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x, y), s\right) \geq \begin{cases} \mathcal{N}'\left(S, s\right) & \\ \mathcal{N}'\left(S\left\{||x||^R + ||y||^R\right\}, s\right), & R \neq 3, 4; \\ \mathcal{N}'\left(S\left\{||x||^R||y||^R + \left\{||x||^{2R} + ||y||^{2R}\right\}\right\}, s\right), & 2R \neq 3, 4; \end{cases} \tag{131}$$

for all $x, y \in \mathcal{B}_1$, then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \begin{cases} \mathcal{N}'\left(\left[2^3 + k^4\right] S, s\left[\lambda|2^3 - 1| + 2|k^4 - 1|\right]\right), \\ \mathcal{N}'\left(\left[2^3 + k^4\right] S||x||^R, s\left[\lambda|2^3 - 2^R| + 2|k^4 - k^R|\right]\right), \\ \mathcal{N}'\left(\left[2^3 + k^4\right] S||x||^{2R}, s\left[\lambda|2^3 - 2^{2R}| + 2|k^4 - k^{2R}|\right]\right), \end{cases} \tag{132}$$

for all $x \in \mathcal{B}_1$.

5.3. Radu's method.

Theorem 5.7. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (89), for all $x, y \in \mathcal{B}_1$ and all $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ with the condition*

$$\lim_{N\to\infty} \mathcal{N}'\left(L(\ell_I^N x, \ell_I^N y), \ell_I^{3N}\right) = 1 \tag{133}$$

where

$$\ell_I = \begin{cases} 2 & if \quad I = 0, \\ \frac{1}{2} & if \quad I = 1 \end{cases} \tag{134}$$

holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function

$$L(x,0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right)$$

satisfies the following property

$$\mathcal{N}'\left(\frac{1}{\ell_I^3} L(\ell_I x, 0), s\right) = \mathcal{N}'\left(\mathcal{L} L(x,0), s\right) \tag{135}$$

for all $x \in \mathcal{B}_1$ and $s > 0$. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}'\left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}} L(x,0), s\right), \tag{136}$$

for all $x \in \mathcal{B}_1$.

Proof. Consider the set

$$\mathcal{A} = \{f/f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2,\ f(0) = 0\}$$

and introduce the generalized metric $d : \mathcal{A} \times \mathcal{A} \to [0, \infty]$ as follows:

$$d(f,g) = \inf\{\omega \in (0,\infty) : \mathcal{N}\left(f(x) - g(x), s\right) \geq \mathcal{N}'\left(\omega\ L(x,0), s\right), x \in \mathcal{B}_1, s > 0\}. \tag{137}$$

It is easy to show that $(\mathcal{A}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $U : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Uf(x) = \frac{1}{\ell_I^3} f_a(\ell_I x),$$

for all $x \in \mathcal{B}_1$. For given $f, f_a \in \mathcal{A}$ and

$$\mathcal{N}\left(f(x) - f_a(x), s\right) \geq \mathcal{N}'\left(\omega\ L(x,0), s\right), x \in \mathcal{B}_1 \text{ and all } s > 0.$$

So, we have

$$\begin{aligned}
\mathcal{N}\left(f(x) - f_a(x), s\right) &= \mathcal{N}\left(\frac{1}{\ell_I^3} f(\ell_I x) - \frac{1}{\ell_I^3} f_a(\ell_I x), s\right) \\
&\geq \mathcal{N}'\left(\frac{\omega}{\ell_I^3} L(\ell_I x, 0), s\right) \\
&= \mathcal{N}'\left(\mathcal{L}\omega L(x,0), s\right)
\end{aligned}$$

for all $x \in \mathcal{B}_1$ and $s > 0$, that is,

$$d(Uf, Uf_a) \leq \mathcal{L} d(f, f_a), \qquad \text{for all} \qquad f, f_a \in \mathcal{A}.$$

This implies U is a strictly contractive mapping on $\mathcal{A}$ with Lipschitz constant $\mathcal{L}$.

For the case $I = 0$, it follows from (137),(97) and (135), we reach

$$\mathcal{N}\left(Uf(x) - f(x), s\right) \geq \mathcal{N}'\left(\mathcal{L}\ L(x,0), s\right), \quad x \in \mathcal{B}_1, s > 0. \tag{138}$$

Hence,

$$d(Uf, f) \leq \mathcal{L}^{1-0}, \qquad f \in \mathcal{A}. \tag{139}$$

For the case $I = 1$, it follows from (137),(111) and (135), we get

$$\mathcal{N}\left(f\left(x\right) - Uf(x), s\right) \geq \mathcal{N}'\left(L(x, 0), s\right), \qquad x \in \mathcal{B}_1, s > 0. \tag{140}$$

Thus, we obtain

$$d(f, Uf) \leq \mathcal{L}^{1-1}, f \in \mathcal{A}. \tag{141}$$

Hence, from (139) and (141), we arrive

$$d(Uf, f) \leq \mathcal{L}^{1-I}, \qquad f \in \mathcal{A}, \tag{142}$$

where $I = 0, 1$. Hence, property (FP1) holds. It follows from property (FP2) that there is a fixed point $\mathcal{C}$ of U in $\mathcal{A}$ such that

$$\mathcal{C}(x) = \lim_{N\to\infty} \frac{1}{\ell_I^{3N}} f(\ell_I^{3N} x), \tag{143}$$

for all $x \in \mathcal{B}_1$. In order to show that $\mathcal{C}$ satisfies (1), the proof is similar lines to that of 5.1 By property (FP3), $\mathcal{C}$ is the unique fixed point of U in the set

$$\Delta = \{\mathcal{C} \in \mathcal{A} : d(f, \mathcal{C}) < \infty\},$$

such that

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}'\left(\omega L(x, 0), s\right), \qquad x \in \mathcal{B}_1, s > 0.$$

Finally, by property (FP4), we obtain

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}'\left(f(x) - Uf(x), s\right), \qquad x \in \mathcal{B}_1, s > 0.$$

This implies

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}'\left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}, s\right)$$

which yields

$$\mathcal{N}\left(f(x) - \mathcal{C}(x), s\right) \geq \mathcal{N}\left(\left(\frac{\mathcal{L}^{1-i}}{1-\mathcal{L}}\right) L(x, 0), s\right), x \in \mathcal{B}_1, s > 0.$$

So, the proof is completed. □

Using Theorem 5.7, we prove the following corollary concerning the stabilities of (1).

Corollary 5.8. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y),s\right) \geq \begin{cases} \mathcal{N}'\left(S,s\right) & \\ \mathcal{N}'\left(S\left\{||x||^R+||y||^R\right\},s\right), & R\neq 3; \\ \mathcal{N}'\left(S\left\{||x||^R||y||^R+\left\{||x||^{2R}+||y||^{2R}\right\}\right\},s\right), & 2R\neq 3; \end{cases} \tag{144}$$

for all $x,y \in \mathcal{B}_1$, then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x)-\mathcal{C}(x),s\right) \geq \begin{cases} \mathcal{N}'\left(2^3\ S, s\ \lambda|2^3-1|\right), \\ \mathcal{N}'\left(2^3\ S||x||^R, s\ \lambda|2^3-2^R|\right), \\ \mathcal{N}'\left(2^3\ S||x||^{2R}, s\ \lambda|2^3-2^{2R}|\right) \end{cases} \tag{145}$$

for all $x \in \mathcal{B}_1$.

Proof. Let

$$L\left(x,y\right) = \begin{cases} S; \\ S\left(||x||^R+||y||^R\right); \\ S\left(||x||^R||y||^R+||x||^{2R}+||y||^{2R}\right); \end{cases}$$

for all $x,y \in \mathcal{B}_1$. Now

$$\begin{aligned} \mathcal{N}'\left(L(\ell_I^N x, \ell_I^N y), \ell_I^{3N} s\right) &= \begin{cases} \mathcal{N}'\left(S, \ell_I^{3N} s\right) \\ \mathcal{N}'\left(S\left\{||\ell_I^N x||^R+||\ell_I^N y||^R\right\}, \ell_I^{3N} s\right), \\ \mathcal{N}'\Big(S\Big\{||\ell_I^N x||^R\ ||\ell_I^N y||^R \\ \qquad\qquad +\left\{||\ell_I^N x||^{2R}+||\ell_I^N y||^{2R}\right\}\Big\}, \ell_I^{3N} s\Big) \end{cases} \\ &= \begin{cases} \rightarrow 1 \text{ as } N\rightarrow\infty, \\ \\ \rightarrow 1 \text{ as } N\rightarrow\infty, \\ \\ \rightarrow 1 \text{ as } N\rightarrow\infty. \end{cases} \end{aligned}$$

Thus, (133) holds. But, we have

$$L\left(x,0\right) = \frac{2^3}{\lambda} L\left(\frac{x}{2},0\right)$$

for all $x \in \mathcal{B}_1$. With the property

$$\mathcal{N}'\left(\frac{1}{\ell_I^3} L(\ell_I x, 0), s\right) = \mathcal{N}'\left(\mathcal{L}L(x,0), s\right).$$

Hence,

$$\mathcal{N}'\left(L(x,0),s\right)=\mathcal{N}'\left(\frac{2^3}{\lambda}L\left(\frac{x}{2},0\right),s\right)=\begin{cases}\mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda},s\right),\\ \mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right),\\ \mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda\cdot 2^{2R}}||x||^{2R},s\right)\end{cases} \tag{146}$$

for all $x\in\mathcal{B}_1$. It follows from (146),

$$\mathcal{N}'\left(\frac{1}{\ell_I^3}L(\ell_I x,0),s\right)=\begin{cases}\mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda},\ell_I^3 s\right),\\ \mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda}||x||^R,\ell_I^{3-R}s\right)\\ \mathcal{N}'\left(\dfrac{S\ 2^3}{\lambda}||x||^{2R},\ell_I^{3-2R}s\right).\end{cases}$$

Hence, the inequality (136) holds for

$$\mathcal{L}=\ell_I^3=\begin{cases}2^3, & 2^{3-R}, & 2^{3-2R} & if \quad I=0,\\ \frac{1}{2^3}, & \frac{1}{2^{3-R}}, & \frac{1}{2^{3-2R}} & if \quad I=1.\end{cases}$$

Now, from (136), we prove the following cases for condition (i).

$$\begin{aligned}&\mathcal{N}\left(f(x)-\mathcal{C}(x),s\right)\\ &\quad\geq\mathcal{N}'\left(\left(\tfrac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right)L(x,0),s\right)\\ &\quad=N'\left(\left(\tfrac{(2^{-3})^{1-0}}{1-2^{-3}}\right)\cdot\tfrac{S\ 2^3}{\lambda},s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{2^{-3}}{1-2^{-3}}\right)\cdot\tfrac{S\ 2^3}{\lambda},s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{S\ 2^3}{\lambda(2^3-1)}\right),s\right)\\ &\quad=\mathcal{N}'\left(\left(S\ 2^3\right),s\lambda(2^3-1)\right)\end{aligned}\qquad\begin{aligned}&\mathcal{N}\left(f(x)-\mathcal{C}(x),s\right)\\ &\quad\geq\mathcal{N}'\left(\left(\tfrac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right)L(x,0),s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{(2^3)^{1-1}}{1-2^3}\right)\cdot\tfrac{S\ 2^3}{\lambda},s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{1}{1-2^3}\right)\cdot\tfrac{S\ 2^3}{\lambda},s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{S\ 2^3}{\lambda(1-2^3)}\right),s\right)\\ &\quad=\mathcal{N}'\left(\left(S\ 2^3\right),s\lambda(1-2^3)\right)\end{aligned}$$

Also, from (136), we prove the following cases for condition (ii).

$$\begin{aligned}&\mathcal{N}\left(f(x)-\mathcal{C}(x),s\right)\\ &\quad\geq\mathcal{N}'\left(\left(\tfrac{\mathcal{L}^{1-i}}{1-\mathcal{L}}\right)L(x,0),s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{(2^{R-3})^{1-0}}{1-2^{R-3}}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{2^{R-3}}{1-2^{R-3}}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{2^R}{2^3-2^R}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(S\ 2^3||x||^R,s\lambda(2^3-2^R)\right)\end{aligned}\qquad\begin{aligned}&\mathcal{N}\left(f(x)-\mathcal{C}(x),s\right)\\ &\quad\geq\mathcal{N}'\left(\left(\tfrac{\mathcal{L}^{1-i}}{1-\mathcal{L}}\right)L(x,0),s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{(2^{3-R})^{1-1}}{1-2^{3-R}}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{1}{1-2^{3-R}}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(\left(\tfrac{2^R}{2^R-2^3}\right)\cdot\tfrac{S\ 2^3}{\lambda\cdot 2^R}||x||^R,s\right)\\ &\quad=\mathcal{N}'\left(S\ 2^3||x||^R,s\lambda(2^R-2^3)\right)\end{aligned}$$

Finally, the proof of (136) for condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

Theorem 5.9. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an odd function fulfilling the inequality (114), for all $x, y \in \mathcal{B}_1$ and $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ with the condition*

$$\lim_{N\to\infty} \mathcal{N}' \left(L(\ell_I^N x, \ell_I^N y), \ell_I^{4N} \right) = 1 \tag{147}$$

where

$$\ell_I = \begin{cases} k & if \quad I = 0, \\ \frac{1}{k} & if \quad I = 1 \end{cases} \tag{148}$$

holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function

$$L\left(x, 0\right) = \frac{1}{k} L\left(\frac{x}{k}, 0\right)$$

with the property

$$\mathcal{N}'\left(\frac{1}{\ell_I^4} L(\ell_I x, 0), s\right) = \mathcal{N}'\left(\mathcal{L} L(x, 0), s\right) \tag{149}$$

for all $x \in \mathcal{B}_1$ and all $s > 0$. Then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \mathcal{N}'\left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}} L(x, 0), s\right), \tag{150}$$

for all $x \in \mathcal{B}_1$.

Proof. The proof of the theorem is similar lines to that of Theorem 5.7 by defining

$$Uf(x) = \frac{1}{\ell_I^4} f_a(\ell_I x),$$

for all $x \in \mathcal{B}_1$. □

Using Theorem 5.9, we prove the following corollary concerning the stabilities of (1).

Corollary 5.10. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x, y), s\right) \geq \begin{cases} \mathcal{N}'\left(S, s\right) & \\ \mathcal{N}'\left(S\left\{||x||^R + ||y||^R\right\}, s\right), & R \neq 4; \\ \mathcal{N}'\left(S\left\{||x||^R||y||^R + \left\{||x||^{2R} + ||y||^{2R}\right\}\right\}, s\right), & 2R \neq 4; \end{cases} \tag{151}$$

for all $x, y \in \mathcal{B}_1$, then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \begin{cases} \mathcal{N}'\left(k^4\ S, 2\ s|k^4 - 1|\right), \\ \mathcal{N}'\left(k^4\ S||x||^R, 2\ s|k^4 - k^R|\right), \\ \mathcal{N}'\left(k^4\ S||x||^{2R}, 2\ s|k^4 - k^{2R}|\right) \end{cases} \tag{152}$$

for all $x \in \mathcal{B}_1$.

Theorem 5.11. *Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a function fulfilling the inequality (124), for all $x, y \in \mathcal{B}_1$ and all $s > 0$, where $L : \mathcal{B}_1^2 \longrightarrow [0, \infty)$ with the conditions (133) and (147) where ℓ_I are respectively defined in (134) and (148) holds for all $x, y \in \mathcal{B}_1$. Assume that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function*

$$L(x,0) = \frac{2^3}{\lambda} L\left(\frac{x}{2}, 0\right) \quad \textit{and} \quad L(x,0) = \frac{1}{k} L\left(\frac{x}{k}, 0\right)$$

with the property

$$\mathcal{N}'\left(\frac{1}{\ell_I^3} L(\ell_I x, 0), s\right) = \mathcal{N}'\left(\mathcal{L}L(x,0), s\right), \ \mathcal{N}'\left(\frac{1}{\ell_I^4} L(\ell_I x, 0), s\right) = \mathcal{N}'\left(\mathcal{L}L(x,0), s\right) \tag{153}$$

for all $x \in \mathcal{B}_1$ and $s > 0$.. Then there is only one cubic mapping $\mathcal{C} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ and only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \mathcal{N}'\left(\left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right)\left(L(x,0) + L(-x,0)\right), s\right) \tag{154}$$

for all $x \in \mathcal{B}_1$.

Using Theorem 5.11, we prove the following corollary concerning the stabilities of (1).

Corollary 5.12. *Assume that S and R are positive numbers. Let $f : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be an even function fulfilling the inequality*

$$\mathcal{N}\left(F_k(x,y), s\right) \geq \begin{cases} \mathcal{N}'\left(S, s\right) & \\ \mathcal{N}'\left(S\left\{||x||^R + ||y||^R\right\}, s\right), & R \neq 3, 4; \\ \mathcal{N}'\left(S\left\{||x||^R||y||^R + \left\{||x||^{2R} + ||y||^{2R}\right\}\right\}, s\right), & 2R \neq 3, 4; \end{cases} \tag{155}$$

for all $x, y \in \mathcal{B}_1$, then there is only one quartic mapping $\mathcal{Q} : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ satisfying the functional equation (1) and

$$\mathcal{N}\left(f(x) - \mathcal{Q}(x), s\right) \geq \begin{cases} \mathcal{N}'\left(\left[2^3 + k^4\right] S, s\left[\lambda|2^3 - 1| + 2|k^4 - 1|\right]\right), \\ \mathcal{N}'\left(\left[2^3 + k^4\right] S||x||^R, s\left[\lambda|2^3 - 2^R| + 2|k^4 - k^R|\right]\right), \\ \mathcal{N}'\left(\left[2^3 + k^4\right] S||x||^{2R}, s\left[\lambda|2^3 - 2^{2R}| + 2|k^4 - k^{2R}|\right]\right), \end{cases} \tag{156}$$

for all $x \in \mathcal{B}_1$.

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Pedagogical Department E.E., Section of Mathematics and Informatics,, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece.

Email address: jrassias@primedu.uoa.gr

Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

Email address: drarun4maths@gmail.com

Department of Mathematics, Shanmuga Industries Arts and Science College, Tiruvannamalai - 606 603, TamilNadu, India.

Email address: sathya24mathematics@gmail.com