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Cohen's factorization theorem for ternary Banach algebras

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Abstract. In this paper, we prove Cohen's factorization theorem for ternary Banach algebras.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians such as A. Cayley [3] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([8]). The comments on physical applications of ternary structures can be found in [1, 9, 10, 12, 13].

A nonempty set G with a ternary operation $[.,.,.] : G \times G \times G \longrightarrow G$ is called a ternary groupoid and denoted by $(G, [.,.,.])$. The ternary groupoid $(G, [.,.,.])$ is called a ternary semigroup if the operation $[.,.,.]$ is associative, i.e., if

$$[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$$

holds for all $x, y, z, u, v \in G$. A ternary semigroup $(G, [.,.,.]))$ is a ternary group if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[x, a, b] = [a, y, b] = [a, b, z] = c,$$

which the elements x, y, z are uniquely determined (see [11]).

A ternary Banach algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \longmapsto [x, y, z]$ of A^3 into A, which is associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [y, z, u], v]$, and satisfy $\|[x, y, z]\| \leq \|x\|\|y\|\|z\|$. An element $e \in A$ is an identity of A if $x = [x, e, e] = [e, e, x]$ for all $x \in A$.

For ternary Banach algebra A, a set $U \times V$ is said to be an approximating set for A (U and V are bounded subsets of A) if for every $\epsilon > 0$, and every $a \in A$, there exist $u \in U$,$v \in V$ such that $\|[u, v, a] - a\| < \epsilon$, $\|[u, a, v] - a\| < \epsilon$, $\|[a, u, v] - a\| < \epsilon$. In [7], the authors proved that the existing of an approximating set for a ternary

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Banach algebra A, implies existence of a bounded approximate identity for it ([7, Theorem 2.1]), in the other words, Altman's Theorem has been proved for ternary case. Some results on special derivations and homomorphisms are obtained in [5, 6].

Assume that A is a ternary Banach algebra, a bounded net (e_α, f_α) is a left bounded approximate identity for A if $\lim_\alpha[e_\alpha, f_\alpha, a] = a$ for all $a \in A$. Similarly, a bounded net (e_α, f_α) is a right bounded approximate identity for A if $\lim_\alpha[a, e_\alpha, f_\alpha] = a$ for all $a \in A$. Also, (e_α, f_α) is a middle bounded approximate identity for A if $\lim_\alpha[e_\alpha, a, f_\alpha] = a$ for all $a \in A$. A net (e_α, f_α) is a bounded approximate identity for A if (e_α, f_α) is a left, right and middle bounded approximate identity for A. If ternary Banach algebra A has a left and right bounded approximate identity, then it has a bounded approximate identity (see [7, Theorem 2.2]).

Let A be a Banach ternary algebra and X be a Banach space. Then X is called a ternary Banach A-module, if module operations $A \times A \times X \longrightarrow X$, $A \times X \times A \longrightarrow X$, and $X \times A \times A \longrightarrow X$ which are $\mathbb{C}$-linear in every variable. Moreover satisfy

(1) $\big[[x, a, b]_X c, d\big]_X = \big[x, [a, b, c]_A, d\big]_X = \big[x, a, [b, c, d]_A\big]_X$,
(2) $\big[[a, x, b]_X, c, d\big]_X = \big[a, [x, b, c]_X, d\big]_X = \big[a, x, [b, c, d]_A\big]_X$,
(3) $\big[[a, b, x]_X, cd, \big]_X = \big[a, [b, x, c]_X, d\big]_X = \big[a, b, [x, c, d]_X\big]_X$,
(4) $\big[[a, b, c]_A, x, d\big]_X = \big[a, [b, c, x]_X, d\big]_X = \big[a, b, [c, x, d]_X\big]_X$,
(5) $\big[[a, b, c]_A, d, x\big]_X = \big[a, [b, c, d]_A, x\big]_X = \big[a, b, [c, d, x]_X\big]_X$,

for every $x \in X$ and all $a, b, c, d \in A$. Obviously, the ternary algebra A is a ternary A-module. A bounded approximate identity in A for X is a bounded net (e_α, f_β) in A such that $\lim_\alpha[x, e_\alpha, f_\alpha] = x$, $\lim_\alpha[e_\alpha, x, f_\alpha] = x$ and $\lim_\alpha[e_\alpha, f_\alpha, x] = x$ for all $x \in X$. For binary Banach algebra A, and for a fixed positive $\epsilon > 0$, if the Banach algebra A has a bounded approximate identity for X then every element $x \in$ can be written as $x = ay$ where $a \in A$ and $y \in X$, and $\|x - y\| < \epsilon$ (Cohen's factorization theorem, see [2, Theorem 10], pp. 61 or [4, Theoerm 2.9.24]). We prove the ternary version of Cohen's factorization theorem for ternary Banach algebras with a different method.

2. Main Results

Let A be a ternary (complex) Banach algebra without identity. Then $A^\sharp$ is the linear space $A \times \mathbb{C}$, where $(A \times \mathbb{C}) \times (A \times \mathbb{C}) \times (A \times \mathbb{C}) \longrightarrow (A \times \mathbb{C})$ or $A^\sharp \times A^\sharp \times A^\sharp \longrightarrow A^\sharp$ together with

$$\big((a, \alpha), (b, \beta), (c, \gamma)\big) \longmapsto \big[(a, \alpha), (b, \beta), (c, \gamma)\big]_{A^\sharp}$$

which is associative in the sense that

$$\begin{aligned}\big[[(a, \alpha), (b, \beta), (c, \gamma)]_{A^\sharp}, (x, \lambda), (y, \mu)\big]_{A^\sharp} &= \big[(a, \alpha), (b, \beta), [(c, \gamma), (x, \lambda), (y, \mu)]_{A^\sharp}\big]_{A^\sharp} \\ &= \big[(a, \alpha), [(b, \beta), (c, \gamma), (x, \lambda)]_{A^\sharp}, (y, \mu)\big]_{A^\sharp},\end{aligned}$$

where $(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta)$ for every $a, b, c, x, y, \in A$ and $\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}$. We denote the identity of $A^\sharp$ by $e(= (0, 1))$, and we write $a + \alpha e$ and a for the elements $[(a,\alpha),(0,1),(0,1)]_{A^\sharp}$ and $[(a,0),(0,1),(0,1)]_{A^\sharp}$ of $A^\sharp$, respectively. By easy calculation one can show that $A^\sharp$ satisfies

$$\|[(a,\alpha),(b,\beta),(c,\gamma)]_{A^\sharp}\| \leq (\|a\|+|\alpha|)(\|b\|+|\beta|)(\|c\|+|\gamma|).$$

Now; define $A := \big\{[(x,0),(y,0),(z,0)] \mid x,y,z \in A\big\}$. Then

$$[(a,\alpha),(b,\beta),[(x,0),(y,0),(z,0)]]$$

is in A. By the above argued statements, we have the following result:

Proposition 2.1. *Every non-unital ternary Banach algebra can be embedded in a unital ternary Banach algebra.*

Now, we prove the main result of paper, which can be regarded as Cohen's factorization theorem for ternary Banach algebras.

Theorem 2.2. *Let $\mathcal{A}$ be a ternary Banach algebra and X be a ternary Banach A-module. If A possess a bounded approximate identity for X, then for all $x \in X$ and each $\epsilon > 0$, there exist $a \in A$ and $y \in X$ such that $x = ay$ and $\|x-y\| < \epsilon$.*

Proof. Let (e_α, f_α) be a bounded approximate identity for A, bounded by $C > 1$. Choose the positive numbers γ and β which satisfy the following conditions:

$$0 < \frac{\gamma}{1+\gamma} < \frac{1}{2C}, \quad \text{and} \quad 1 < \beta < 1+\gamma. \tag{1}$$

Let a be an arbitrary element in A such that $\|z\| \leq 1$. The above mentioned conditions imply that $\frac{1}{C^n}\|[e_\alpha, f_\alpha, z]\|^n < 1$, for $n \geq 1$. Therefore

$$(2^n(\gamma(1+\gamma)^{-1})^n\|[e_\alpha,f_\alpha,z]\|^n) < \frac{1}{C^n}\|[e_\alpha,f_\alpha,z]\|^n < 1,$$

and thereby we have

$$(\gamma(1+\gamma)^{-1})^n\|[e_\alpha,f_\alpha,z]\|^n < \frac{1}{2^n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Now, suppose that $S_n = \sum_{i=0}^{n}(\gamma(1+\gamma)^{-1}[e_\alpha,f_\alpha,z])^i$. Then

$$\begin{aligned}\|S_{n+1}-S_n\| &= \|(\gamma(1+\gamma)^{-1})^n[e_\alpha,f_\alpha,z]^n\| \leq (\gamma(1+\gamma)^{-1})^{n+1}\|[e_\alpha,f_\alpha,z]\|^{n+1} \\ &\leq \frac{1}{2^{n+1}} \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow \infty$. This means that (S_n) is a Cauchy sequence. Thus, the series $\sum_{i=0}^{\infty}(\gamma(1+\gamma)^{-1}[e_\alpha,f_\alpha,z])^i$ converges in A. Then $([e,e,e]+\gamma[e,e,e]-\gamma[e_\alpha,f_\alpha,e])$ is invertible in $A^\sharp$, and we have

$$\begin{aligned}([e,e,e]+\gamma[e,e,e]-\gamma[e_\alpha,f_\alpha,e])^{-1} &= (1+\gamma)^{-1}([e,e,e]-\frac{\gamma}{1+\gamma}[e_\alpha,f_\alpha,e])^{-1} \\ &= (1+\gamma)^{-1}\sum_{i=0}^{\infty}(\gamma(1+\gamma)^{-1}[e_\alpha,f_\alpha,e])^i.\end{aligned}$$

So,

$$\|([e,e,e]+\gamma[e,e,e]-\gamma[e_\alpha,f_\alpha,e])^{-1}\| \leq \sum_{i=0}^{\infty}\frac{\gamma(1+\gamma)^{-1}}{2^n} < 2. \tag{2}$$

Assume that $[e_n,f_n,e]=[e_{\alpha_n},f_{\alpha_n},e]$ such that $\|\gamma[e,e,x]-\gamma[e_n,f_n,x]\|<\epsilon/2^n$, $n\in\mathbb{N}$. Define $t_n=([e,e,e]+\gamma[e,e,e]-\gamma[e_1,f_1,e])\cdots([e,e,e]+\gamma[e,e,e]-\gamma[e_n,f_n,e])$. Since every $([e,e,e]+\gamma[e,e,e]-\gamma[e_j,f_j,e])$ is invertible for $1\leq j\leq n$, t_n is invertible. Now, set $a_n=t_n^{-1}-([e,e,e]+\gamma[e,e,e])^{-n}\in A$ and $y_n=[t_n,e,x]_X$. Choose an element $(e_{n+1},f_{n+1})\in A\times A$ such that

$$\|[e_{n+1},f_{n+1},a_n]-a_n\|<\frac{1}{\beta^n} \quad \text{and} \quad \|\gamma[t_n,e,x]_X-\gamma[t_n,e_{n+1},f_{n+1}x]_X\|_X<\frac{1}{2^{n+1}}. \tag{3}$$

By definition of a_n, relations (1), (2) and (3), we have

$$\begin{aligned}
\|a_{n+1}-a_n\| &= \|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}a_n \\
&\quad +\Big(([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}-([e,e,e]+\gamma[e,e,e])^{-1}\Big) \\
&\quad \times([e,e,e]+\gamma[e,e,e])^{-n} \\
&\quad -([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}[e_{n+1},f_{n+1},a_n] \\
&\quad +([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}[e_{n+1},f_{n+1},a_n] \\
&\qquad -[e_{n+1},f_{n+1},a_n]+[e_{n+1},f_{n+1},a_n]-a_n\| \\
&\leq \|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}\|\|a_n-[e_{n+1},f_{n+1},a_n]\| \\
&\quad +\|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}-([e,e,e]+\gamma[e,e,e])^{-1}\| \\
&\quad \times\|([e,e,e]+\gamma[e,e,e])^{-n}\| \\
&\quad +\|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}-[e,e,e]\|\|[e_{n+1},f_{n+1},a_n]\| \\
&\quad +\|[e_{n+1},f_{n+1},a_n]-a_n\| \\
&< \frac{2}{\beta^n}+\frac{M}{\beta^n}+\frac{N}{\beta^n}+\frac{1}{\beta^n}\longrightarrow 0 \ \text{ as } \ n\longrightarrow\infty,
\end{aligned} \tag{4}$$

where $M=\|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}-([e,e,e]+\gamma[e,e,e])^{-1}\|$ and $N=\|([e,e,e]+\gamma[e,e,e]-\gamma[e_{n+1},f_{n+1},e])^{-1}-[e,e,e]\|$. Then by (4), we conclude that (a_n) is a Cauchy sequence. Therefore there exists an element $a\in A$ such that $a=\lim_n a_n$ (it is clear that $t_n^{-1}\longrightarrow a$). By the above obtained results it is easy to see that (y_n) is a Cauchy sequence in X. Thereupon, there exists $y\in X$ such that $y=\lim_n y_n$. By gathering the obtained results, we have $x=ay$ and $\|x-y\|<\epsilon$. □

Corollary 2.3. *Let $\mathcal{A}$ be a ternary Banach algebra with a left bounded approximate set. Then, for all $a\in A$ and each $\epsilon>0$, there exist $b,c\in A$ such that $a=bc$ and $\|a-c\|<\epsilon$.*

References

[1] V. Abramov, R. Kerner, O. Liivapuu and S. Shitov, *Algebras with ternary law of composition and their realization by cubic matrices*, *Journal of Generalized Lie Theory and Applications*, 3(2)(2009), 77–94.

[2] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer–Verlag, Berlin, 1973.

[3] A. Cayley, *On the* 34 *concomitants of the ternary cubic*, Amer. J. Math., 4(1-4)(1881), 1–15.

[4] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Society Monographs, 24, Clarendon Press, Oxford, 2000.

[5] M. Eshaghi Gordji, R. Farrokhzad and S. A. R. Hosseinioun, *Hyers-Ulam stability of ternary* (σ, t, ζ)*-derivations on* C^**-ternary algebras*, J. Math. Phys. Anal. Geom., 8(1)(2012), 3-20.

[6] M. Eshaghi Gordji, A. Jabbari, A. Ebadian and S. Ostadbashi, *Automatic continuity of 3-homomorphisms on ternary Banach algebras*, Inter. J. Geom. Meth. Mod. Phys., 10(10)(2013), 1320013.

[7] M. Eshaghi Gordji, A. Jabbari and G. H. Kim, *Approximate identity in ternary Banach algebras*, Abs. Appl. Anal., (2012), Article ID 386785, 6 pages.

[8] M. Kapranov, I. M. Gelfand and A. Zelevinskii, *Discrimininants*, Resultants and Multidimensional Determinants, Birkhauser, Berlin, 1994.

[9] R. Kerner, *Ternary and non-associative structures*, Inter. J. Geom. Meth. Modern Phys., 5(8)(2008), 1265–1294.

[10] R. Kerner, *The cubic chessboard*, Geom. Phys. Class. Quantum Grav., 14(1A)(1997), 203-225.

[11] E. L. Post, *Polyadic groups*, Trans. Amer. Math. Soc., 48(1940), 208-350.

[12] G. L. Sewell, *Quantum mechanics and its emergent macrophysics*, Princeton Univ. Press, Princeton, NJ, 2002.

[13] H. Zettl, *A characterization of ternary rings of operators*, Adv. Math., 48(1993), 117-143.

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