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Stability of cosine type functional equations on module extension Banach algebras

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Abstract. Let A be a Banach algebra and X be a Banach A-bimodule. In this paper we investigate the stability of the cosine type functional equation

$$\varphi(ab, a\cdot y+x\cdot b)+\varphi(ab, x\cdot b-a\cdot y)=2\varphi(a,x)\varphi(b,y),$$

on module extension Banach algebra $\mathfrak{A}=A\oplus X$.

1. Introduction

The problem of the stability of functional equations was originally stated by Ulam in [10]. He proposed the following famous question concerning the stability of homomorphisms:

Let G be a group and let G' be a metric group with metric d. Given $\varepsilon>0$, does there exist $\delta>0$ such that if a $f: G\longrightarrow G'$ satisfies

$$d(f(xy), f(x)f(y))<\delta \qquad x,y\in G,$$

then there is a homomorphism $F: G\longrightarrow G'$ with $d(f(x),F(x))<\varepsilon$, for all $x\in G$?

In [5], Hyers considered the case of approximately additive mappings $f: X\longrightarrow Y$, where X and Y are Banach spaces and f satisfies

$$\|f(x+y)-f(x)-f(y)\|\leq\varepsilon,$$

for all $x,y\in X$. It was shown that the limit

$$F(x)=\lim_{n\to\infty}2^{-n}f(2^n x),$$

exists for all $x\in X$ and that $F: X\longrightarrow Y$ is the unique additive mapping satisfying

$$\|f(x)-F(x)\|\leq\varepsilon.$$

A generalization of Hyers' theorem provided by Th. M. Rassias [8].

The functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{1}$$

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is called the cosine or *d'Alembert's* functional equation [6].

The stability of the functional equation (1) was investigated by Baker in [2]. He proved the following.

Let $\varepsilon \geq 0$ be a given number and G be an abelian group. Then any unbounded solution $f : G \longrightarrow \mathbb{C}$ of the inequality

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon,$$

satisfies the cosine equation (1). Badora and Ger [1] proved its superstability under the condition

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \text{ or } \varphi(y).$$

The stability of Cosine functional equation is also investigated in [3, 4] and it generalized by Kannappan and Kim [7].

Suppose that $\mathfrak{A} = A \oplus X$, where A is a Banach algebra and X is a Banach A-bimodule. Then $\mathfrak{A}$ with norm $\|(a,x)\| = \|a\| + \|x\|$ and product

$$(a,x)(b,y) = (ab, a \cdot y + x \cdot b) \quad (a, b \in A,\ x, y \in X),$$

is a Banach algebra which is known as a module extension Banach algebra.

The map $T : \mathfrak{A} \longrightarrow \mathfrak{A}$, defined by $T(a,x) = (a,-x)$, is multiplicative. That is, $T((a,x)(b,y)) = T(a,x)T(b,y)$, and $T \circ T = id$, where id is the identity map. Let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a function and consider the functional equation

$$\varphi(ab, a \cdot y + x \cdot b) + \varphi(ab, x \cdot b - a \cdot y) = 2\varphi(a,x)\varphi(b,y). \tag{2}$$

By setting $p = (a,x)$ and $q = (b,y)$ in (2), we obtain

$$\varphi(pq) + \varphi(pT(q)) = 2\varphi(p)\varphi(q), \quad p, q \in \mathfrak{A}. \tag{3}$$

This equation has the same form as the cosine functional equation (1).

We say that a function $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is of approximate a cosine type function, if there is $\delta > 0$ such that

$$|\varphi(pq) + \varphi(pT(q)) - 2\varphi(p)\varphi(q)| < \delta, \quad p, q \in \mathfrak{A}.$$

In the case where $\delta = 0$, φ satisfies the functional equation (3).

A Banach A-bimodule X is said to be unit linked if A has a unit element e and $e \cdot x = x \cdot e = x$, for all $x \in X$, and it is called symmetric if $a \cdot x = x \cdot a$, for all $a \in A, x \in X$. For example, each commutative Banach algebra A is symmetric A-bimodule over itself.

The main purpose of this paper is to prove the stability problem of equation (3), on the module extension Banach algebra $\mathfrak{A}$.

2. Stability of equation (3)

In the following result we assume that X is a symmetric Banach A-bimodule and $\mathfrak{A} = A \oplus X$.

Theorem 2.1. *Let $f, g : A \longrightarrow [0, \infty)$ be functions and $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ satisfies the functional inequality*

$$|\varphi(pq) + \varphi(pT(q)) - 2\varphi(p)\varphi(q)| \leq min\{f(a), g(b)\}. \tag{4}$$

Then the map $F : A \longrightarrow \mathbb{C}$ given by $F(t) = \varphi(t, 0)$ for all $t \in A$, is either bounded or multiplicative. Moreover, φ satisfies in

$$|\varphi(p)^2 - \frac{1}{2}\varphi(p^2) - \frac{1}{2}F(a^2)| \leq \frac{1}{2}\ min\{f(a), g(a)\}, \quad p \in \mathfrak{A}.$$

Proof. Setting $p = (a, 0)$, $q = (b, 0)$ in (4), we get

$$|\varphi(ab, 0) - \varphi(a, 0)\varphi(b, 0)| \leq \frac{1}{2}\ min\{f(a), g(b)\},$$

for all $a, b \in A$. Thus, it follows from [9] that $F(a) = \varphi(a, 0)$ for all $a \in \mathcal{A}$, is either bounded or multiplicative. Now setting $p = q = (a, x)$, in (4), we get

$$|\varphi(a^2, 2(a \cdot x)) + \varphi(a^2, 0) - 2\varphi(a, x)^2| \leq min\{f(a), g(a)\}.$$

Therefore

$$|\varphi(p)^2 - \frac{1}{2}\varphi(p^2) - \frac{1}{2}F(a^2)| \leq \frac{1}{2}\ min\{f(a), g(a)\},$$

as requerd. □

Proposition 2.2. *Suppose that A is a unital commutative Banach algebra, $\varphi : A \oplus A \longrightarrow \mathbb{C}$ satisfies the functional inequality (4), and let $h(x) = \varphi(e, x)$ for all $x \in A$. Then*

(1) *h is either bounded, or*
(2) *h satisfies the cosine functional equation*

$$h(x + y) + h(x - y) = 2h(x)h(y), \quad x, y \in A.$$

Proof. Setting $p = (e, x)$, $q = (e, y)$ in (4). We obtain

$$|\varphi(e, x + y) + \varphi(e, x - y) - 2\varphi(e, x)\varphi(e, y)| \leq\ min\{f(e), g(e)\}.$$

Using $h(x) = \varphi(e, x)$,

$$|h(x + y) + h(x - y) - 2h(x)h(y)| \leq\ min\{f(e), g(e)\}$$

for all $x, y \in A$. Now it follows from Theorem 5 of [2] that h is bounded or it is a cosine function. □

Remark 2.3. *We recall that by Theorem 2 of* [2], *h satisfies in the condition* (2) *of preceding Proposition, that is h satisfies the cosine functional equation if and only if there exists a complex-valued function m defined on A such that for all $a, b \in A$,*

$$h(a) = \frac{1}{2}\big(m(a) + m(-a)\big), \quad \textit{and} \quad m(a+b) = m(a)m(b).$$

Proposition 2.4. *Let A be a commutative Banach algebra, $\varphi : A \oplus A \longrightarrow \mathbb{C}$ satisfies the functional inequality (4). Then φ is either bounded, or $\varphi \circ T = \varphi$.*

Proof. Let $\Gamma_\varphi = \frac{1}{2}(\varphi + \varphi \circ T)$. Since φ satisfies in (4), by the same methoed as in [2] we see that φ is either bounded or

$$\Gamma_\varphi(pq) + \Gamma_\varphi(pT(q)) = 2\Gamma_\varphi(p)\Gamma_\varphi(q), \quad p, q \in A \oplus A.$$

It follows from this equality that $\Gamma_\varphi = \varphi$, and so $\varphi \circ T = \varphi$. □

If we take $q = (0, 0)$ in (4), then we get the following result.

Corollary 2.5. *Let $f, g : A \longrightarrow [0, \infty)$ be functions and let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ with $\varphi(0, 0) \neq 0$, satisfies the functional inequality (4). Then*

$$|\varphi(p) - 1| \leq \frac{1}{2|\varphi(0, 0)|}\; min\{f(a), g(0)\},$$

for each $p = (a, x) \in \mathfrak{A}$, and so φ is bounded.

Corollary 2.6. *Let $\delta > 0$ and $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ with $\varphi(0, 0) \neq 0$, satisfies the functional inequality*

$$|\varphi(pq) + \varphi(pT(q)) - 2\varphi(p)\varphi(q)| \leq \delta. \tag{5}$$

Then φ is bounded and there exist $\lambda \in \mathbb{C}$ such that $|\varphi(p) - 1| \leq \frac{\delta}{2|\lambda|}$, for all $p = (a, x) \in \mathfrak{A}$.

Proof. This follows from Corollary 2.5 for $\lambda = \varphi(0, 0)$ and $f(a) = g(b) = \delta$. □

As a consequence of above Corollary, we obtain the next result.

Corollary 2.7. *Let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a cosine type function. Then $\varphi(p) = 1$ for all $p \in \mathfrak{A}$.*

We denoted by $Inv(A)$, the set of all invertible elements of unital Banach algebra A.

Theorem 2.8. *Suppose that A is a unital commutative Banach algebra, and let $\varphi : A \oplus A \longrightarrow \mathbb{C}$ satisfies the functional inequality (4). If φ is unbounded, then it setisfies in*

$$|\varphi(p) - \frac{1}{2}F(a)(m(xa^{-1}) + m(-xa^{-1}))| \leq \frac{1}{2}min\{f(a), g(e)\}, \tag{6}$$

for each $p = (a, x) \in Inv(A) \oplus A$, where $F : \mathbb{R} \longrightarrow \mathbb{C}$ is a multiplicative function and $m : \mathbb{R} \longrightarrow \mathbb{C}$ is a exponential.

PROOF. By Theorem 2.1, the map $\varphi(a,0) = F(a)$ is a multiplicative function from A to $\mathbb{C}$ and by Proposition 2.2, $\varphi(e,x) = h(x)$ is a solution of the cosine functional equation. It follwes from Remark 2.3, that there exists an exponential functiom $m : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\varphi(e,x) = h(x) = \frac{1}{2}\big(m(x) + m(-x)\big),$$

for all $x \in A$. Taking $p = (a,0)$ and $q = (e, a^{-1}x)$ in (4), gives

$$|\varphi(a,x) + \varphi(a,-x) - 2\varphi(a,0)\varphi(e,xa^{-1})| \leq min\{f(a), g(e)\}, \tag{7}$$

for each $p = (a,x) \in Inv(A) \oplus A$. By Proposition 2.4, $\varphi \circ T = \varphi$. Therefore $\varphi(a,x) = \varphi(a,-x)$, and so inequality (7) implies that (6). □

From Theorem 2.8 we have:

Corollary 2.9. *Let $\delta > 0$, A be a unital commutative Banach algebra, and let $\varphi : A \oplus A \longrightarrow \mathbb{C}$ satisfies the functional inequality (5). If φ is unbounded, then there exists a multiplicative function $F : \mathbb{R} \longrightarrow \mathbb{C}$ and a exponential function $m : \mathbb{R} \longrightarrow \mathbb{C}$ such that*

$$|\varphi(p) - \frac{1}{2}F(a)(m(xa^{-1}) + m(-xa^{-1}))| \leq \frac{\delta}{2},$$

for each $p = (a,x) \in Inv(A) \oplus A$.

Note that if $\delta = 0$ in Corollary 2.9, then for all $p = (a,x) \in Inv(A) \oplus A$, we get

$$\varphi(p) = \frac{1}{2}F(a)(m(xa^{-1}) + m(-xa^{-1})).$$

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References

[1] R. Badora, R. Ger, *On some trigonometric functional inequalities*, Functional Equation Results and Advances, Kluwer Acad. Publ., (2002), 3-15.

[2] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc., 80 (3) (1980), 411-416.

[3] B. Bouikhalene, E. Elqorachi and J. M. Rassias, *The superstability of d'Alembert's functional equation on the Heisenberg group*, Appl. Math. Lett., 23(1) (2010), 105-109.

[4] P. D. P. Friis, *d'Alembert's and Wilson's equations on Lie groups*, Aequa. Math., 67(2004), 12-25.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., 27(1941), 222-224.

[6] Pl. Kannappan, *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc., 19(1968), 69-74.

[7] Pl. Kannappan and G. H. Kim, *On the stability of generalized cosine functional equations*, Ann. Acad. Paedagogicae-Cracoviensis, Studia Math., (2001), 49-58.

[8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(1978), 297-300.

[9] L. Szsekelyhidi, *On a theorem of Baker*, Proc. Amer. Math. Soc., 84(1)(1982), 95-96.

[10] S. M. Ulam, Problems in Modern Mathematics, Chap. VI, Wiley, New York, 1960.

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